The role of extreme points for convex hull operations.

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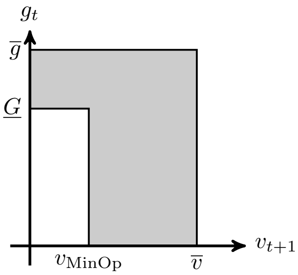
Outline

Modeling non-convex sets

Is there any simple way of modeling the following set?



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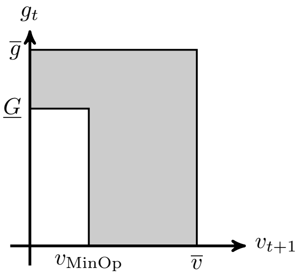
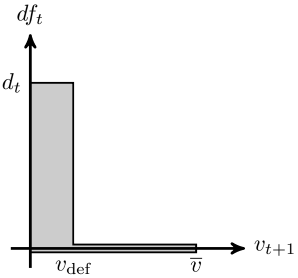


Mathematical formulation:

$$\begin{aligned} (1 - z) \cdot v_{\mathrm{m}} \leq v \leq \overline{v}, \\ z \cdot \underline{g} \leq g \leq \overline{g}, \\ z \in \{0, 1\}. \end{aligned}$$

What about this one?



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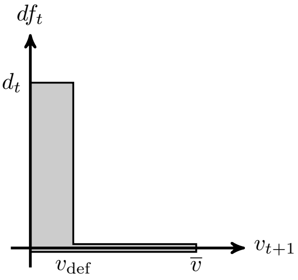
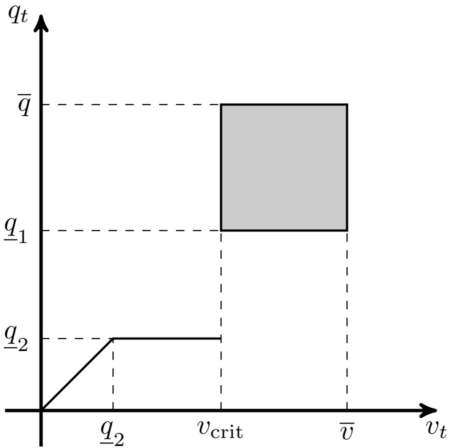


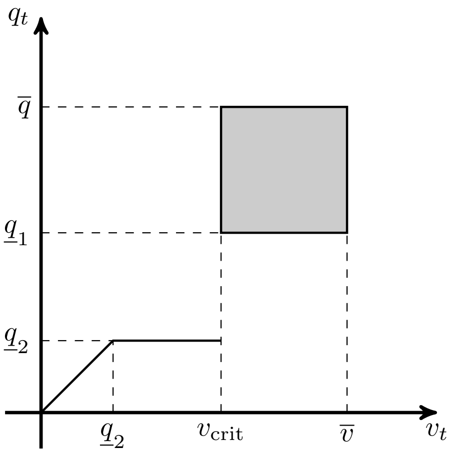
Mathematical formulation:

$$
\begin{aligned}
&0 \leq \ v \ \leq (1-z)\cdot \overline{v} + z \cdot v_{\text{def}},\\
&0 \leq df \leq z \cdot d,\\
&z \in \{0,1\}.
\end{aligned}
$$

How can we formulate this set?



How can we formulate this set? Now it seems harder.



In the general case, our feasible sets are like Tangrans:

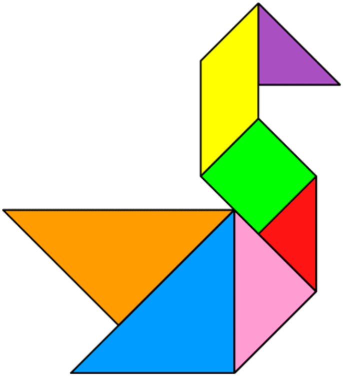


Figure: Tangran feasible set.

Disjunctive Constraints

Union of polyhedra

Let $P_i = \{x \in \mathbb{R}^n \mid A_i x \leq b_i\}$, for $i \in I$. How can we represent the corresponding union $\bigcup_{i \in I} P_i$?

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Balas's formula [Balas, 1979],[Balas, 1998]

The "magic" formula:

$$Q = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} A_i x_i \leq z_i \cdot b_i, \\ \sum_{i \in I} x_i = x,\; \sum_{i \in I} z_i = 1, \\ x_i \in \mathbb{R}^n,\; z_i \in \{0,1\},\; i \in I. \end{array} \right\}$$

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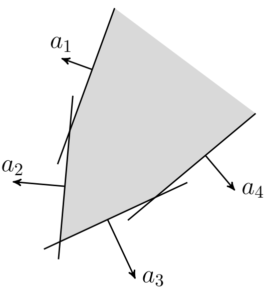
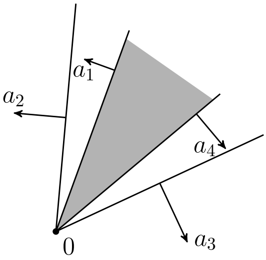
However, the inequality $A_j x_j \le 0$ may have non-zero solutions.

Balas formulas explanation

The following linear constraints provides an example:



(a) $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



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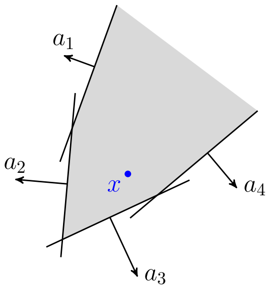

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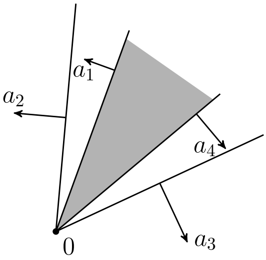

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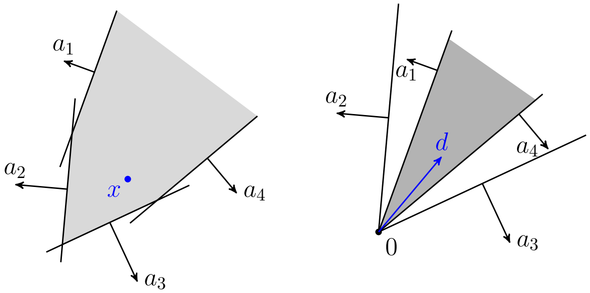

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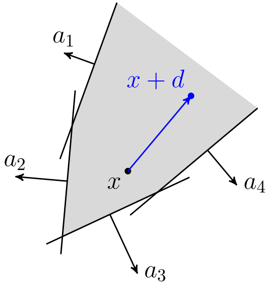

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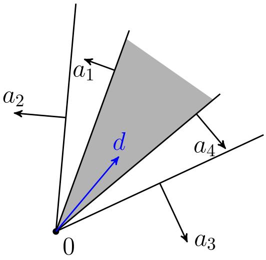

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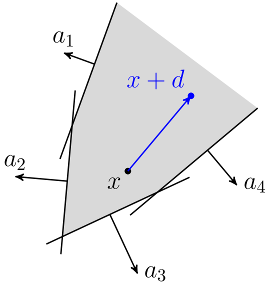

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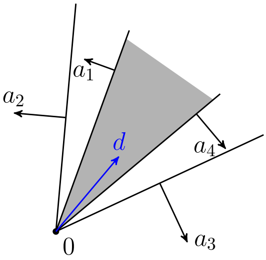

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Then, x belongs to Q if, and only if, there is $x_i \in P_i$ and $d_j \in \text{recc}(P_j)$, such that

$$x = x_i + \sum_{\substack{j=1 \\ j \neq i}}^{r} d_j$$

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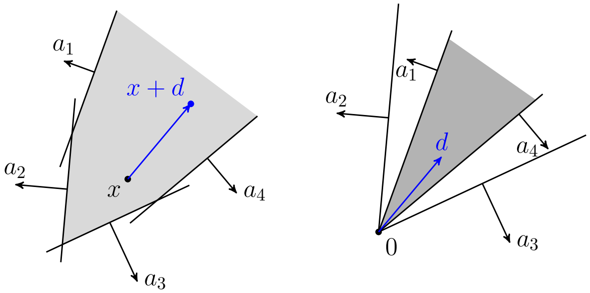

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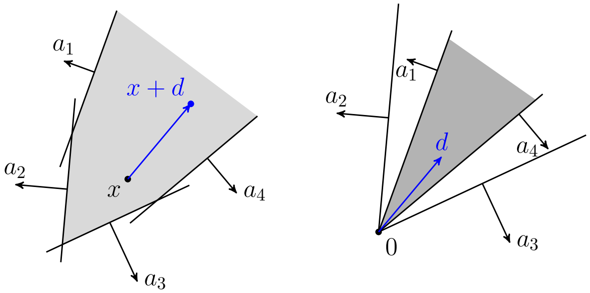

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The set P_i is compact if, and only if, $\text{recc}(P_i) = \{0\}$.

Balas's formula example

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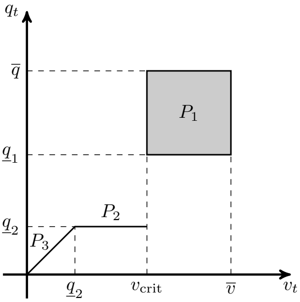

$$P_1 = \{(v,q) \mid \underline{q}_1 \le q \le \overline{q},\ v_{\text{crit}} \le v \le \overline{v}\}$$
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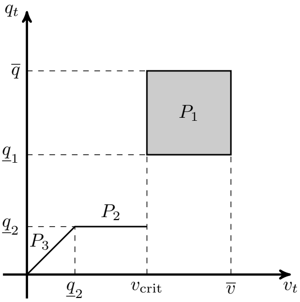

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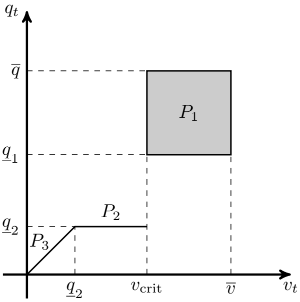

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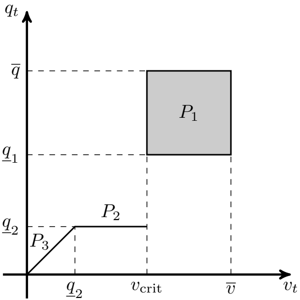

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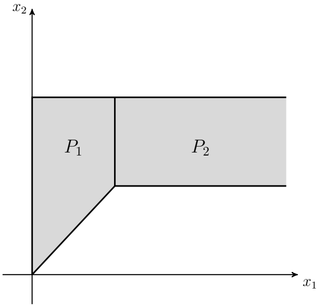
$$

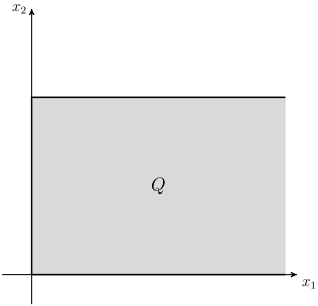
When Balas's formula fails

Theorem (Jeroslow, [Jeroslow, 1987])

If each P_i is non-empty and Balas's formula does not represent $\bigcup_{i=1}^{p} P_i$, then no set of linear constraints involving continuous and binary variables can do it.



(a) Non-representable set.



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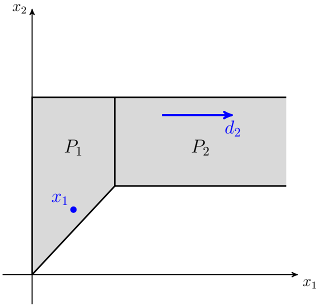

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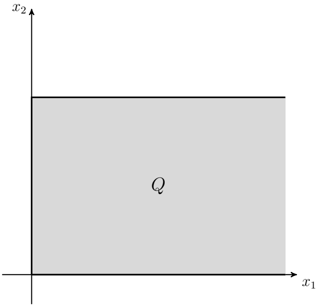

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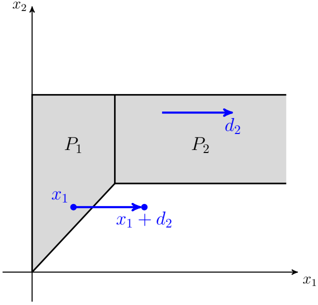

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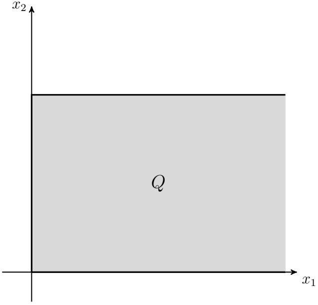

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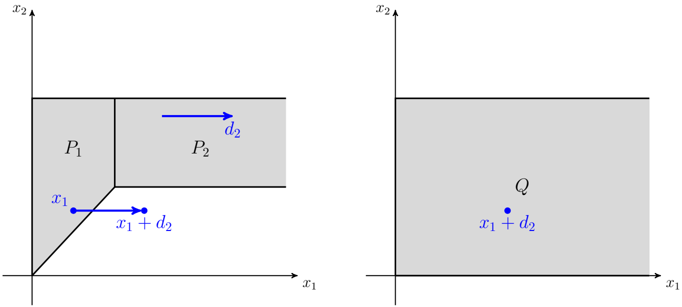

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When solving mixed-integer problems, it is always a good idea to describe the convex hull of the feasible set, since

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where $X \subset \mathbb{R}^n \times \{0,1\}^l$.

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Balas's convex hull Theorem, [Balas, 1998]

If each P_i is non-empty, then the convex hull of $\cup_{i \in I} P_i$ has the same formula as Q but with z_i a continuous variable in $[0,1]$:

$$\operatorname{cl}\operatorname{conv}(\cup_{i\in I} P_i) = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} A_i x_i \le z_i b_i,\ \sum_{i\in I} x_i = x,\ \sum_{i \in I} z_i = 1, \\ x_i \in \mathbb{R}^n,\ z_i \in [0,1],\ i \in I. \end{array} \right\}.$$

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The proof rely on $\operatorname{cl}\operatorname{conv}(\cup_{i\in I} P_i) = \operatorname{conv}(\cup_{i\in I} P_i) + \sum_{i\in I} \operatorname{recc}(P_i)$.

Let us see an illustration for the convex closure formula
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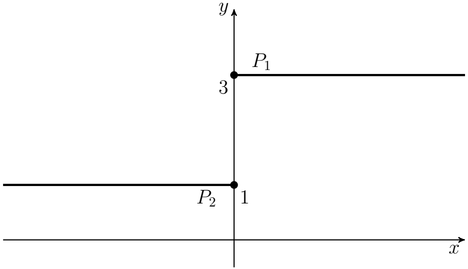

Figure: Convex closure formula for polyhedra.

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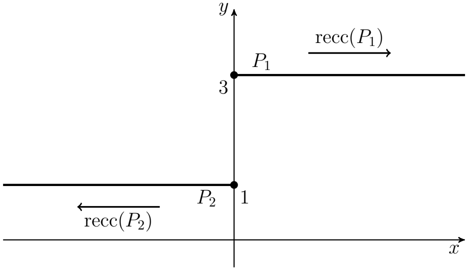

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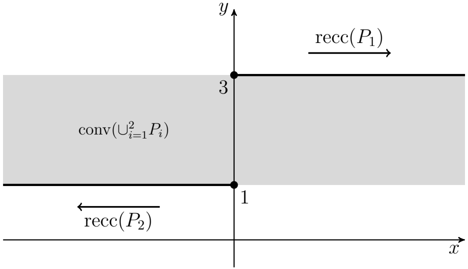

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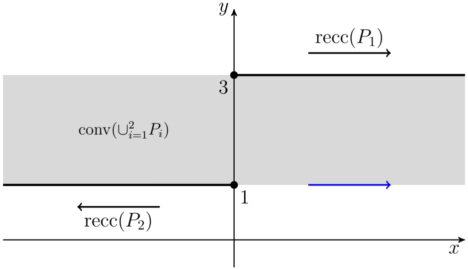

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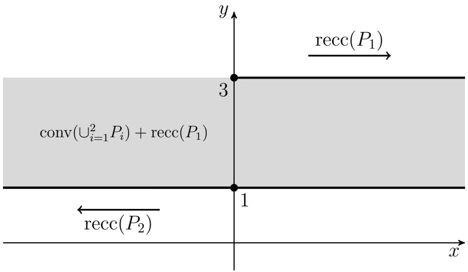

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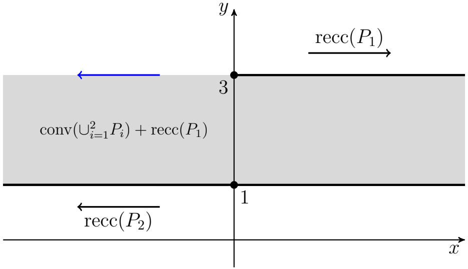

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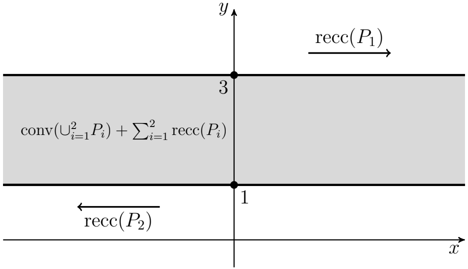

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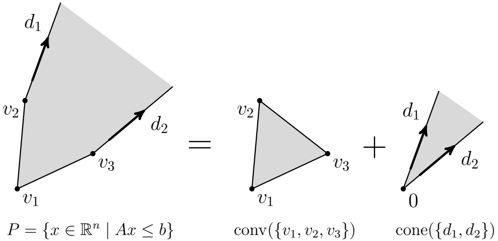
Comments on the convex closure formula

The equation "$\mathrm{cl\,conv}(\cup_{i\in I} P_i) = \mathrm{conv}(\cup_{i\in I} P_i) + \sum_{i\in I} \mathrm{recc}(P_i)$" follows from the Minkowski-Weyl theorem:



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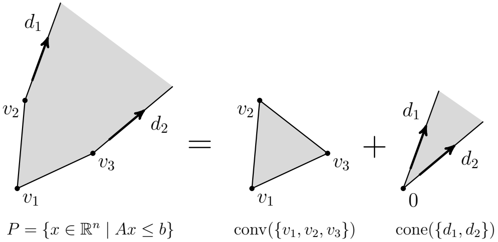

Theorem (Minkowski-Weyl representation)

The set P is polyhedral if and only if there is a finite number of vectors $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ and $\{w_1, \ldots, w_l\} \subset \mathbb{R}^n$ such that

$$P = \operatorname{conv}(\{v_1, \ldots, v_k\}) + \operatorname{cone}(\{w_1, \ldots, w_l\}).$$

In particular, the recession cone of P is equal to $\operatorname{cone}(\{w_1, \ldots, w_l\})$.

From Balas's formula,

$$Q = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} A_i x_i \leq z_i b_i,\ \sum_{i \in I} x_i = x,\ \sum_{i \in I} z_i = 1, \\ x_i \in \mathbb{R}^n,\ z_i \in \{0,1\},\ i \in I. \end{array} \right\},$$

we have seen that

- A given union of polyhedra $\cup_{i \in I} P_i$ is representable by 0-1 mixed integer linear constraints if, and only if, Balas's formula represents it;
- The continuous relaxation of Balas's formula leads to the set $\mathrm{conv}(\cup_{i \in I} P_i) + \sum_{i \in I} \mathrm{recc}(P_i)$;
- The convex closure *always* satisfies the following relation:

$$\mathrm{cl}\,\mathrm{conv}\left(\bigcup_{i \in I} P_i\right) = \mathrm{conv}\left(\bigcup_{i \in I} P_i\right) + \sum_{i \in I} \mathrm{recc}(P_i).$$

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Is it possible to generalize those ideas to general convex sets?

Generalized Disjunctive Constraints

Perspective function

We can rewrite inequality $Ax \leq z \cdot b$ in the following form: $z \cdot [A(x/z) - b] \leq 0$. This motivates the *perspective* function

$$g(x,z) := \begin{cases} z \cdot f(x/z) & \text{if } z > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where f is a proper closed convex function.

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where f is a proper closed convex function. The perspective function g is proper and convex, and if $g(x,1) \leq 0$ then x belongs to the closed convex set D, where

$$D := \{x \in \mathbb{R}^n \mid f(x) \leq 0\}.$$

However, the perspective function g is not defined at $z = 0$.

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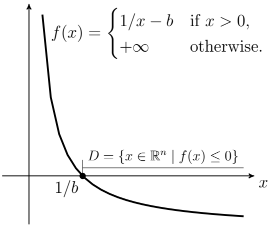
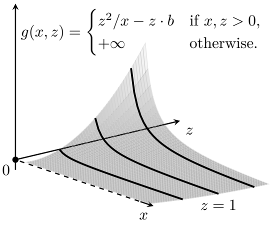
$$D := \{x \in \mathbb{R}^n \mid f(x) \leq 0\}.$$

However, the perspective function g is not defined at $z = 0$. The key to extend g to $z = 0$ is the recession function of f.

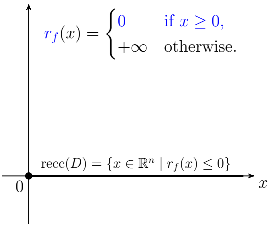
Example



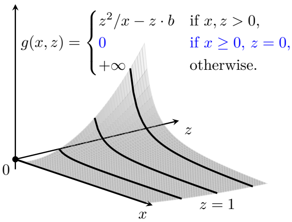
(a) Original function.



(b) Perspective function.



(c) Recession function.



(d) Perspective closure.

Recession function

The recession function r_f is the 'asymptotic slope' of a function f:

$$r_f(d) = \lim_{\alpha \to \infty} \frac{f(x + \alpha d)}{\alpha} = \lim_{z \to 0^+} z \cdot f(x + d/z),$$

$\forall x \in \text{dom}(f)$. We also have that $\text{recc}(D) = \{d \in \mathbb{R}^n \mid r_f(d) \leq 0\}$.

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Theorem (Perspective closure)

The closure of the perspective function g is

$$\overline{g}(x,z) = \begin{cases} z\cdot f(x/z) & \text{if } z>0, \\ r_f(x) & \text{if } z=0, \\ +\infty & \text{otherwise,} \end{cases}$$

which is a proper closed convex function.

In particular, if $g(x,0) \le 0$ then x belongs to $\mathrm{recc}(D)$.

Generalized Balas's formula, [Ceria and Soares, 1999]

Balas's formula for closed convex sets:

$$Q = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \overline{g}_i(x_i, z_i) \le 0, \\ \sum_{i \in I} x_i = x, \; \sum_{i \in I} z_i = 1, \\ x_i \in \mathbb{R}^n, \; z_i \in \{0,1\}, \; i \in I. \end{array} \right\}$$

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$$Q = \bigcup_{i \in I} D_i + \sum_{i \in I} \operatorname{recc}(D_i).$$

If each D_i is compact, then $Q = \bigcup_{i \in I} D_i$.

Theorem (Linear relaxation formula, [Ceria and Soares, 1999])

Consider the Linear Relaxation of the Generalized Balas's formula,

$$\overline{Q} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \overline{g}_i(x_i, z_i) \le 0, \\ \sum_{i \in I} x_i = x, \ \sum_{i \in I} z_i = 1, \\ x_i \in \mathbb{R}^n, \ z_i \in [0,1], \ i \in I. \end{array} \right\}.$$

Then, the set $\overline{Q}$ *is equal to* $\operatorname{conv}\left(\bigcup_{i \in I} D_i\right) + \sum_{i \in I} \operatorname{recc}(D_i)$.

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The proof is identical to the polyhedral case.

However, the convex closure of $\bigcup_{i \in I} D_i$ may be different from $\overline{Q}$:

$$\mathrm{cl}\,\mathrm{conv}\left(\bigcup_{i \in I} D_i\right) \supset \mathrm{conv}\left(\bigcup_{i \in I} D_i\right) + \sum_{i \in I} \mathrm{recc}(D_i).$$

Counterexample for equality:

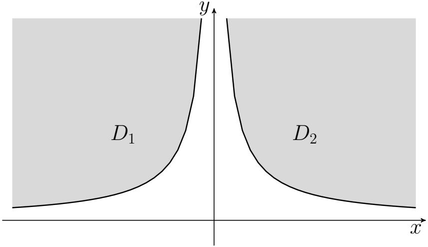

Figure: Convex closure counterexample.

Theorem (Regularity conditions for convex closure formula)

Let $D_1, \ldots, D_m$ be nonempty closed convex sets. If for all choice of directions $d_i \in \mathrm{recc}(D_i)$ which sum to zero, $d_1 + \cdots + d_m = 0$, their opposite directions $-d_i$ also belong to $\mathrm{recc}(D_i)$, then the convex closure formula holds.

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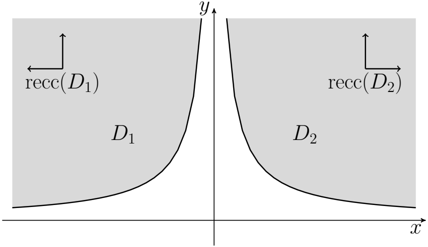

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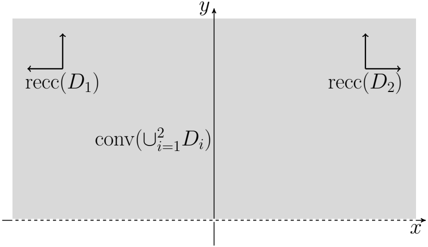

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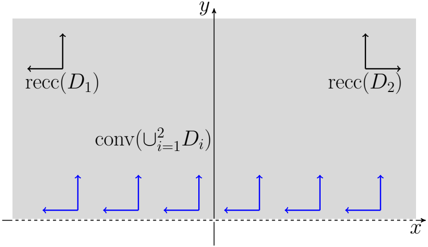

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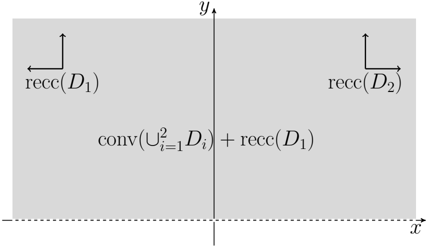

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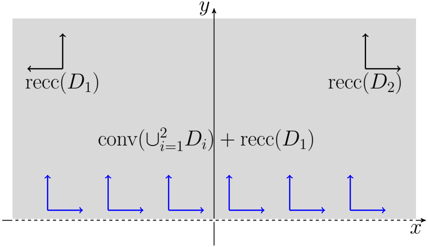

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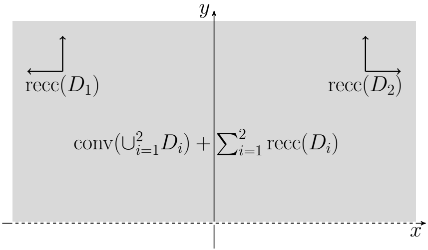

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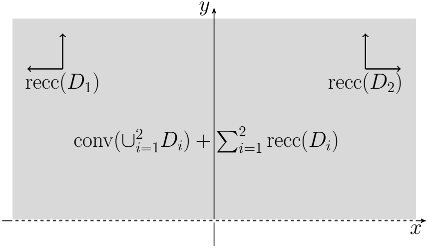

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Corollary (Lack of extreme points)

Let $D_1, \ldots, D_m$ be nonempty closed convex sets. If the convex closure formula does not hold, then $\text{cl}\,\text{conv}(\bigcup_{i=1}^m D_i)$ has no extreme point.

Summary

From the Generalized Balas's formula,

$$Q = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \overline{g}(x_i, z_i) \leq 0, \; \sum_{i \in I} x_i = x, \; \sum_{i \in I} z_i = 1, \\ x_i \in \mathbb{R}^n, \; z_i \in \{0,1\}, \; i \in I. \end{array} \right\},$$

we have seen that

- The closure of the perspective function provides the suitable framework for the Balas's formula on general convex sets;
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Is it possible to understand more geometrically the Balas's formula and the corresponding continuous relaxation?

Blessing of Extreme Points

Balas's formula

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Balas's theorem revisited

Balas's formula is the projection of the following set

$$Q_{\text{lift}} = \left\{ (x,z) \in \mathbb{R}^{n+|I|} \;\middle|\; \begin{array}{c} A_i x_i \leq z_i b_i, \\ \sum_{i\in I} x_i = x,\ \sum_{i \in I} z_i = 1, \\ x_i \in \mathbb{R}^n,\ z_i \in \{0,1\},\ i \in I. \end{array} \right\}.$$

Balas's theorem revisited

Ext & Cvx Hull
F. Cabral
Modeling non-convex sets
Disjunctive Constraints
Generalized Disjunctive Constraints
Blessing of Extreme Points
Example: SDDiP
Take away

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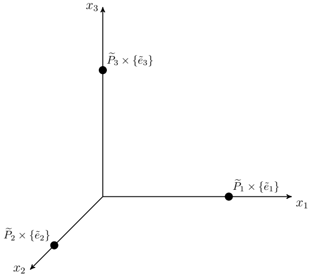
Suppose that each P_i is compact. Then, $Q_{\text{lift}} = \cup_{i \in I}(P_i \times \{e_i\})$.



(a) Lifted Balas set (pictorial)

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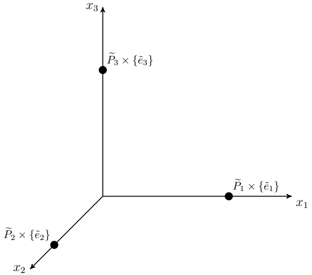

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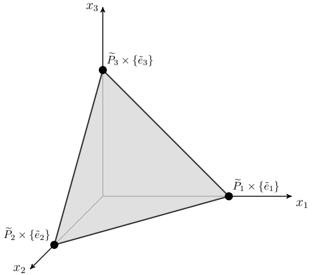
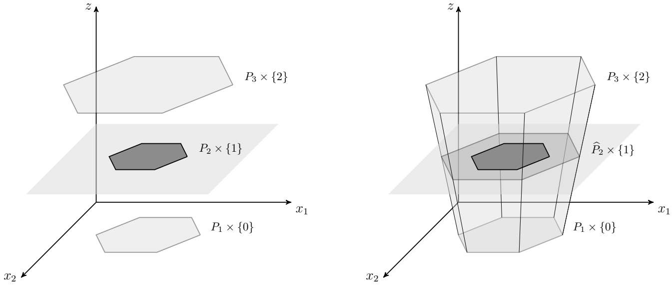


(b) Lifted Balas convex closure

Why is the simplex lift so special? What if we had used another geometry?



(a) Non-extremal lift (b) Loss of information

Figure: Non-extremal lifted set and the corresponding convex closure.

Blessing of Extreme Points – discrete version

Theorem (Blessing of extreme points – discrete version)

Let $\{D_i\}_{i \in I}$ be convex sets and $\{r_i\}_{i \in I}$ be extreme points. Then,

$$\text{conv}\left(\cup_{i \in I}(D_i \times \{r_i\})\right) \cap \left(\mathbb{R}^n \times \{r_j\}\right) = D_j \times \{r_j\}, \quad \text{for all } j.$$

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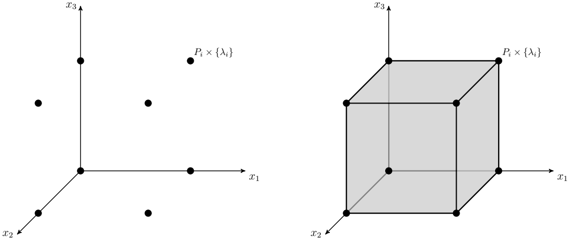

(a) Extremal lift (b) Information preserved

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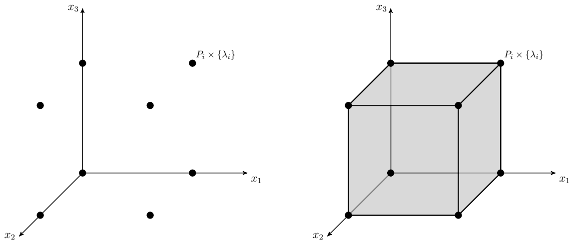
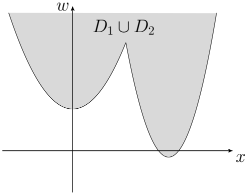


(a) Extremal lift (b) Information preserved

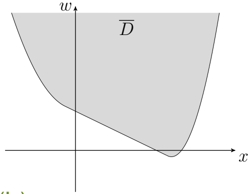
Cartesian product with extreme points are preserved under convex hull operation.

Blessing of Extreme Points – continuous version

Let D_1 and D_2 be two closed convex sets and $\overline{D} = \mathrm{cl}\,\mathrm{conv}(\cup_{i=1}^{p} D_i)$.



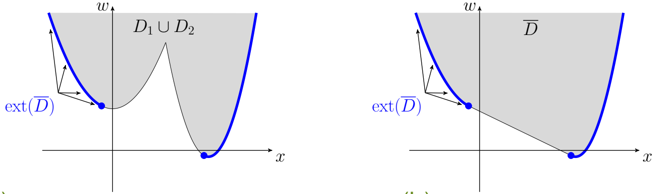
(a) Union of closed convex sets



(b) Convex closure.

Blessing of Extreme Points – continuous version

Let D_1 and D_2 be two closed convex sets and $\overline{D} = \mathrm{cl\,conv}(\cup_{i=1}^{p} D_i)$.



(a) Union of closed convex sets (b) Convex closure.

Theorem (Blessing of Extreme Points – set version)

Let $\{D_i\}_{i \in I}$ be nonempty closed convex sets. Then,

$$\mathrm{ext}\left(\mathrm{cl\,conv}\left(\bigcup_{i \in I} D_i\right)\right) \subseteq \bigcup_{i \in I} \mathrm{ext}(D_i). \tag{1}$$

Blessing of Extreme Points – continuous version

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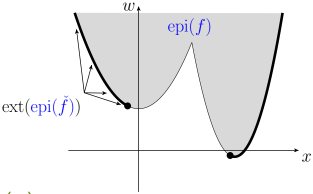
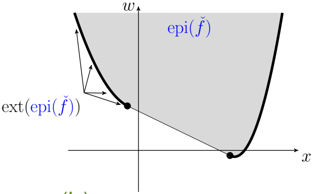


(a) Union of closed convex sets



(b) Convex closure.

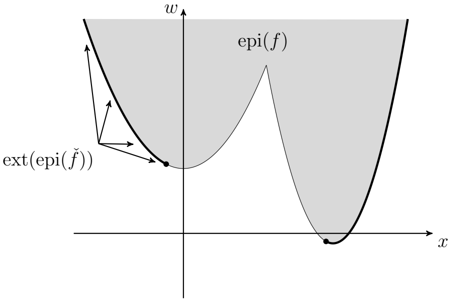
Corollary (Blessing of Extreme Points – function version)

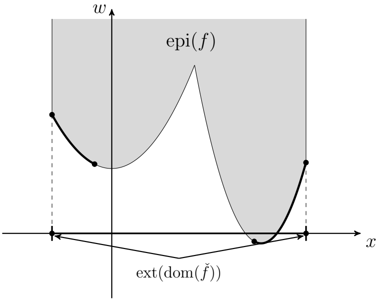
Let f be the minimum of a finite number of proper closed convex functions. Then, the convex regularization $\check{f}$ satisfies to

$$\check{f}(\bar{x}) = f(\bar{x}),$$

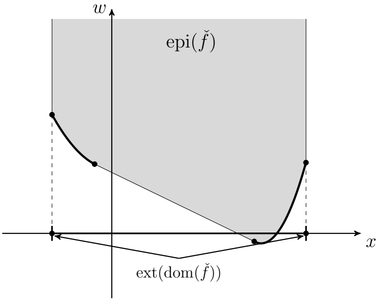
for all $\bar{x} \in \mathbb{R}^n$ such that $(\bar{x}, \check{f}(\bar{x}))$ is an extreme point of $\operatorname{epi}(\check{f})$.

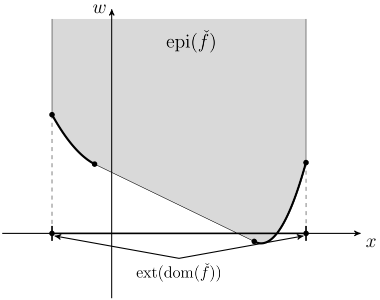
Blessing of Extreme Points




w
epi(f)
x
ext(dom($\check{f}$))

Blessing of Extreme Points





Corollary (Extension of BOB)

Let f be the minimum of a finite number of proper closed convex function. If $\bar{x}$ is an extreme point of $\text{dom}(\check{f})$, then $(\bar{x}, \check{f}(\bar{x}))$ is an extreme point of $\text{epi}(\check{f})$. In particular, $\check{f}(\bar{x}) = f(\bar{x})$.

- Cartesian products with extreme points are preserved by the convex hull operation;
- The extreme points of $\mathrm{cl}(\mathrm{conv}(\cup_{i\in I} D_i))$ belong to the union of extreme sets $\cup_{i\in I}\,\mathrm{ext}(D_i)$;
- The extreme points of $\mathrm{dom}(f)$ have zero convexification gap.

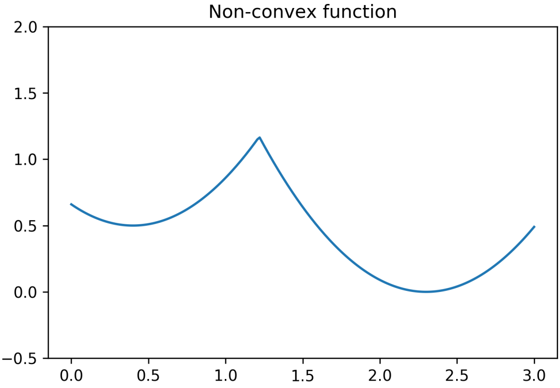
Summary

- Cartesian products with extreme points are preserved by the convex hull operation;
- The extreme points of $\mathrm{cl}(\mathrm{conv}(\cup_{i \in I} D_i))$ belong to the union of extreme sets $\cup_{i \in I} \mathrm{ext}(D_i)$;
- The extreme points of $\mathrm{dom}(f)$ have zero convexification gap.

Let's see how to use those idea to approximate non-convex functions using cutting planes!

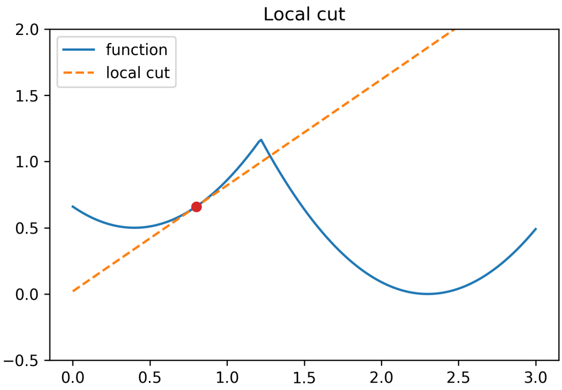
Example: geometrical interpretation of SDDiP

Pictorial representation of a future cost-to-go function in the MILP case.

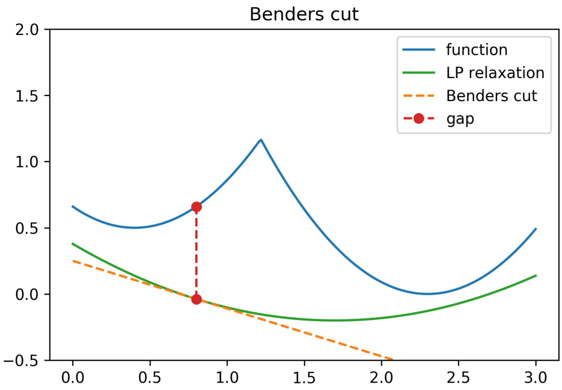


Local cuts

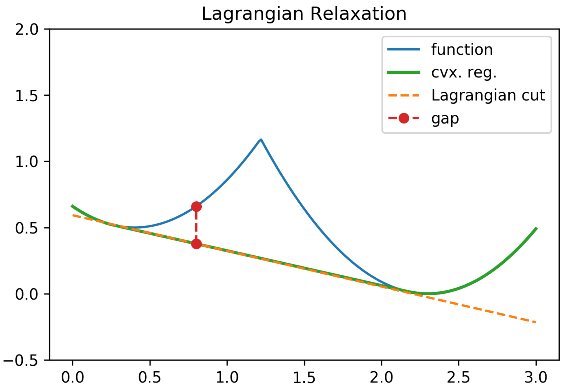
Local cuts are not suitable to get a valid lower bound.



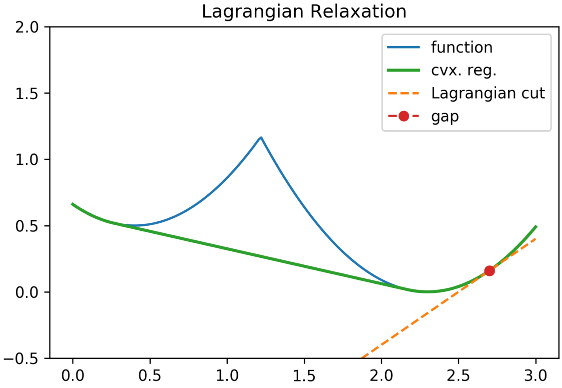
Linear programming relaxation may induce loose cuts.



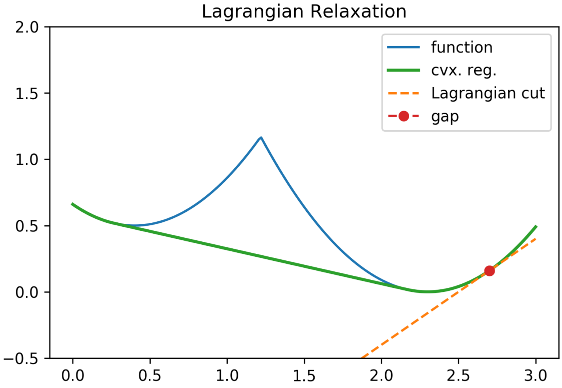
Lagrangian relaxation is the tightest convex approximation, but the induced cut may also have a gap.



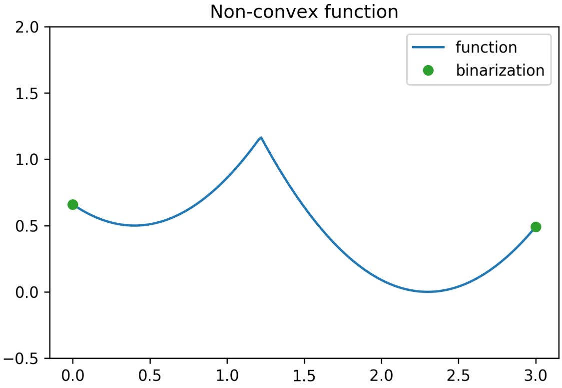
However, there are some special points in which the Lagrangian Relaxation have zero gap.



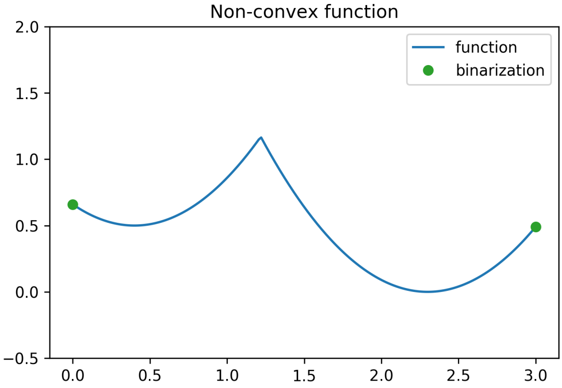
However, there are some special points in which the Lagrangian Relaxation have zero gap.
Can we describe those points?



Note that we only have 2 extreme points in the domain.



Note that we only have 2 extreme points in the domain.
What if we want to preserve additional points?



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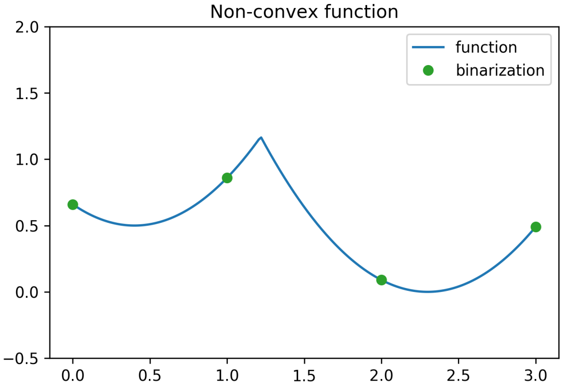

Consider the following change of variables:

$$g(x_0, x_1) = f(x_0 + 2x_1), \quad x_0, x_1 \in [0, 1].$$

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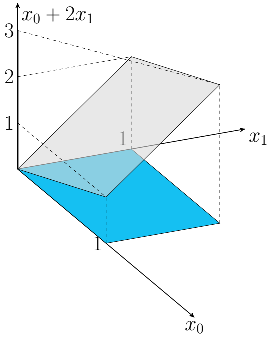

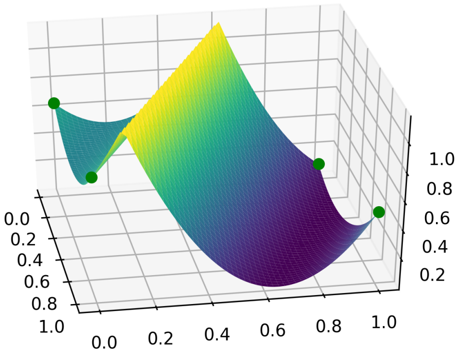

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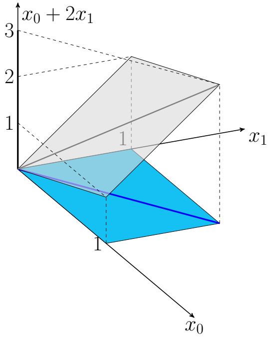

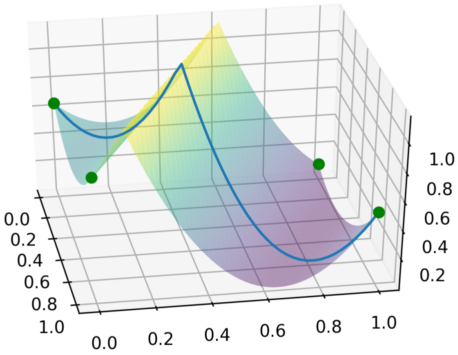

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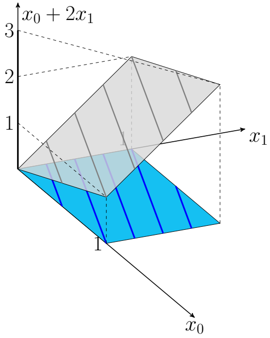

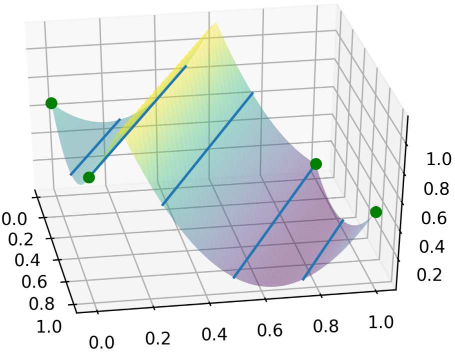

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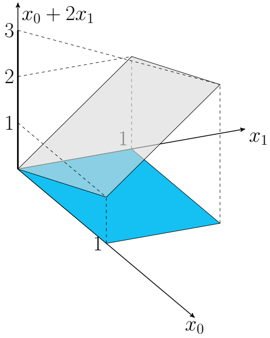

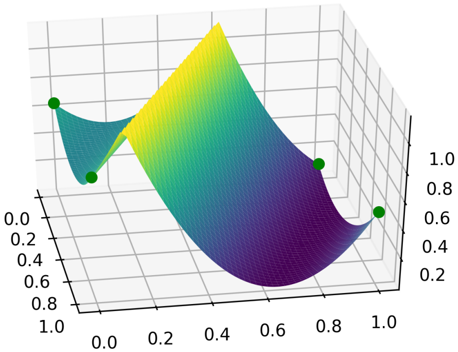

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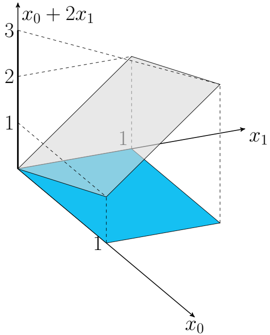

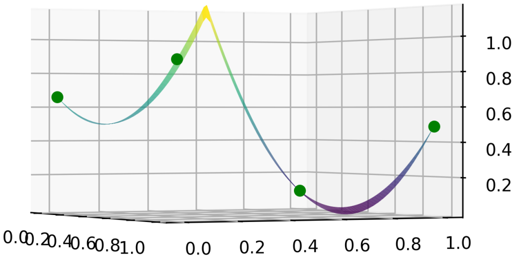

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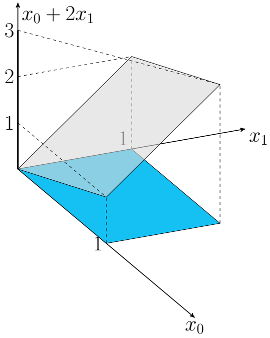

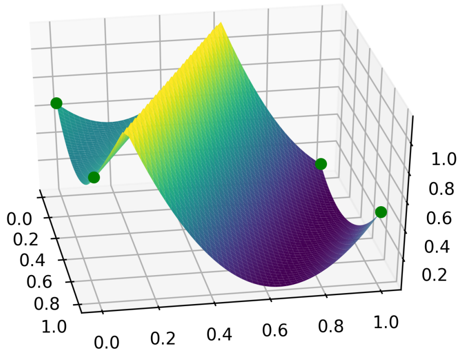

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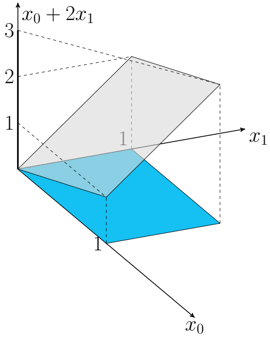
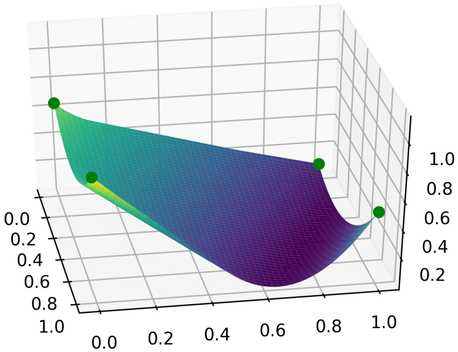


Convex regularization



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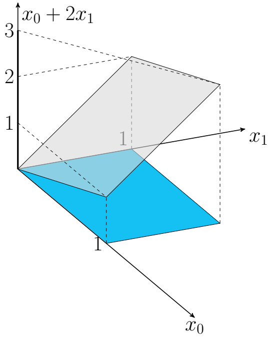
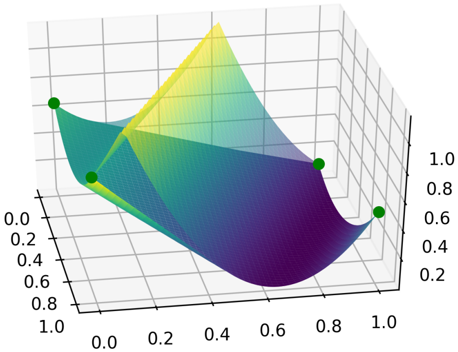
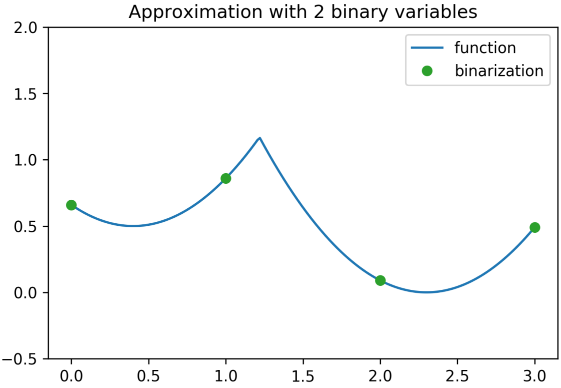
Blessing Of Binary (BOB)!





We could have a lift with $4 = 2^2$ extreme points:

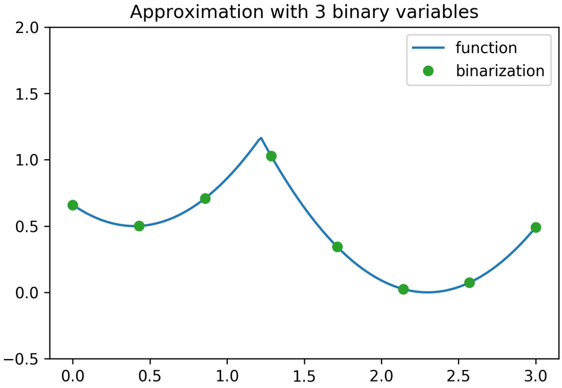
$$x = x_0 + 2x_1.$$



Getting more extreme points

We could have a lift with $8 = 2^3$ extreme points:

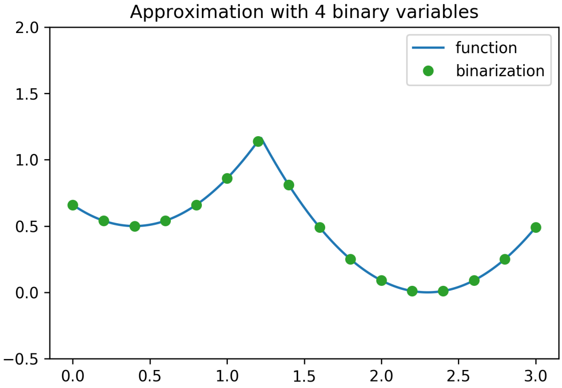
$$x = (x_0 + 2x_1 + 4x_2)\frac{3}{7}.$$



Getting more extreme points

We could have a lift with $16 = 2^4$ extreme points:

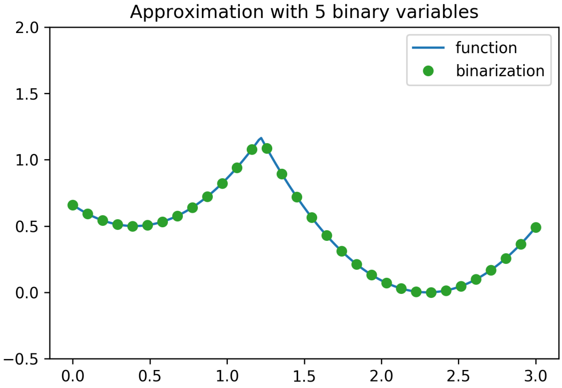
$$x = (x_0 + 2x_1 + 4x_2 + 8x_3)\frac{3}{15}.$$



Getting more extreme points

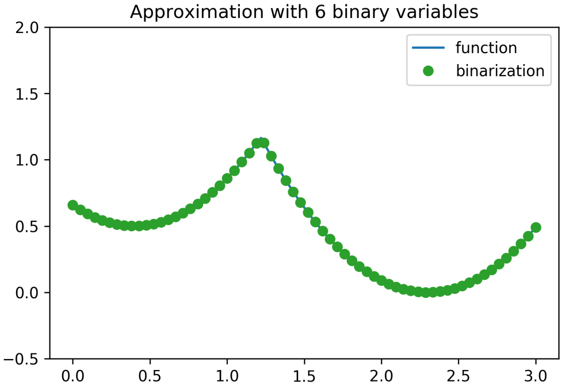
We could have a lift with $32 = 2^5$ extreme points:

$$x = (x_0 + 2x_1 + 4x_2 + 8x_3 + 16x_4)\frac{3}{31}.$$



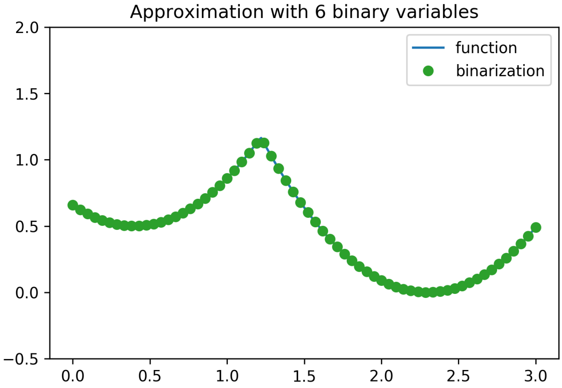
We could have a lift with $64 = 2^6$ extreme points:

$$x = (x_0 + 2x_1 + 4x_2 + 8x_3 + 16x_4 + 32x_5)\frac{3}{63}.$$



We could have a lift with $n = 2^k$ extreme points:

$$x = \epsilon \cdot \sum_{i=0}^{k-1} 2^i x_i.$$



Summary

- Linear approximations of non-convex function are not a valid lower bound or they do have a gap;
- The SDDiP algorithm increases the domain space to create more extreme points, and computes Lagrangian cuts on them.

Take away

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- The Balas formula provides important insights for the relationship between the convex hull and extreme points;
- The Cartesian product of extreme points and convex sets are always preserved by the convex hull operation;
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Thank you!

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