The Stochastic Lipschitz Dynamic Programming (SLDP) algorithm

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Joint work with Shabbir Ahmed (GaTech) and
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In Memoriam: Shabbir Ahmed

The work of professor Shabbir heavily inspired me to get into the Stochastic Mixed Integer Program field, specially the SDDiP algorithm, and I will always keep in my memory our discussions.

Figure: Professor Shabbir Ahmed.

This name is a famous mathematical property in honor of the German mathematician Rudolf Lipschitz.

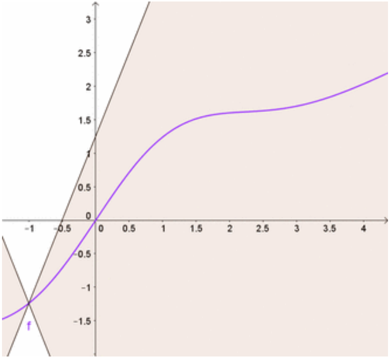


Figure: Lipschitz property illustration.

Introduction

Multistage MILP stochastic program

$$
\begin{aligned}
Q_t(x_{t-1}, h_t) = \quad \min \quad & c_t^\top x_t + \overline{\mathcal{Q}}_t(x_t) \\
s.t. \quad & T_t x_{t-1} + W_t x_t = h_t, \\
& x_t \in \mathbb{R}_+^m \times \mathbb{Z}_+^k,
\end{aligned}
$$

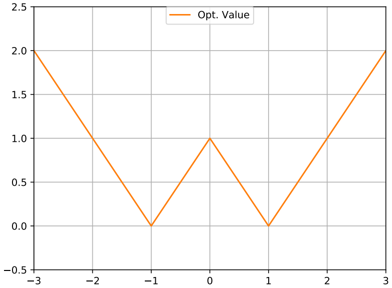
$$
\overline{\mathcal{Q}}_t(x_t) = \begin{cases} \mathbb{E}\left[Q_{t+1}(x_t, h_{t+1})\right] & , t \in \{1, \ldots, T-1\}, \\ 0 & , t = T, \end{cases}
$$

Comments:

- The function Q_t is piecewise linear, but *non-convex*;
- The SLDP algorithm do not require the binarization of the state variables such as the SDDiP [Zou-2019] and also do not assume monotonicity of the cost-to-go function such as the MIDAS [Philpott-2016] algorithm.

Warm up

The function bellow will be used to illustrate the SLDP algorithm:



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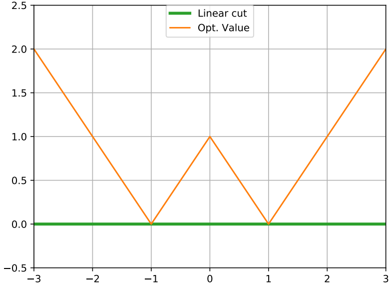

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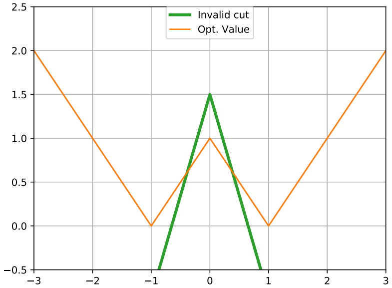

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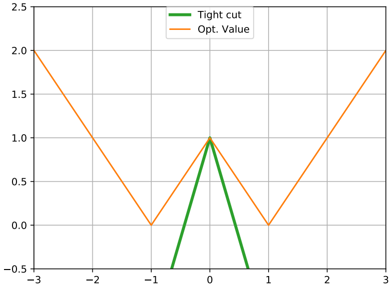

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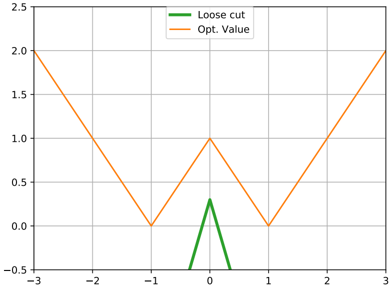

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Let $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be a function. How do we find the vertical translation α so that a given function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ translated by α under-approximates f?

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Nonlinear Lagrangians

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We have a few options for α:

Case	Property on the translation of g
$\alpha < \alpha^*$	$g(x) + \alpha$ is a loose under-estimate for f
$\alpha = \alpha^*$	$g(x) + \alpha$ is the tightest lower approximation of $f(x)$
$\alpha > \alpha^*$	$g(\overline{x}) + \alpha$ is greater than $f(\overline{x})$ for some $\overline{x} \in \mathbb{R}^n$
$\alpha^* = -\infty$	vertical translations of g never under-estimate f

Now, let $\Phi = \{\phi_y(x) \mid y \in Y\}$ be a family of functions paremeterized by Y. We define the Φ-hull of $f(x)$ as the pointwise supremum of all under-approximations $\alpha + \phi_y(x)$:

$$\check{f}(x) = \sup_{\alpha \in \mathbb{R}, y \in Y} \left\{ \alpha + \phi_y(x) \;\middle|\; \begin{array}{r} \alpha + \phi_y(x) \leqslant f(x), \\ \forall x \in \mathbb{R}^n \end{array} \right\}.$$

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We can simplify more this formula. Consider

$$\alpha(y) = \inf_{x \in \mathbb{R}^n} [f(x) - \phi_y(x)],$$

then the Φ-hull of f can be represented as

$$\breve{f}(x) = \sup_{y \in Y} \alpha(y) + \phi_y(x).$$

We call $\alpha(y)$ the Φ-Lagrangian dual function of f.

If we consider the Φ-family

$$\Phi = \left\{ \phi_y(x) \quad \middle| \quad y \in Y \right\},$$

then we have the Φ-Lagrangian dual function

$$\alpha(y) = \inf_{x \in \mathbb{R}^n} f(x) - \phi_y(x) \quad .$$

and the Φ-hull function

$$\check{f}(x) = \sup_{y \in \mathbb{R}^n} \alpha(y) + \phi_y(x) \quad .$$

If we consider the linear-family

$$\Phi = \left\{ y^\top x \;\middle|\; y \in \mathbb{R}^n \right\},$$

then we have the standard-Lagrangian dual function

$$\alpha(y) = \inf_{x \in \mathbb{R}^n} f(x) - y^\top x \qquad .$$

and the convex-hull function

$$\check{f}(x) = \sup_{y \in \mathbb{R}^n} \alpha(y) + y^\top x \qquad .$$

If we consider the proximal-family

$$\Phi = \left\{ -\frac{\rho}{2}\|x-y\|^2 \quad \middle| \; y \in \mathbb{R}^n \right\},$$

then we have the proximal-Lagrangian dual function

$$\alpha(y) = \inf_{x\in\mathbb{R}^n} f(x) + \frac{\rho}{2}\|x-y\|^2 \qquad .$$

and the proximal-hull function

$$\check{f}(x) = \sup_{y\in\mathbb{R}^n} \alpha(y) - \frac{\rho}{2}\|x-y\|^2 \qquad .$$

If we consider the sharp-family

$$\Phi = \left\{ y_\lambda^\top x - \rho \|x - y_w\| \;\middle|\; y \in \mathbb{R}^{n+n} \right\},$$

then we have the sharp-Lagrangian dual function

$$\alpha(y) = \inf_{x \in \mathbb{R}^n} f(x) - y_\lambda^\top x + \rho \|x - y_w\|.$$

and the sharp-hull function

$$\check{f}(x) = \sup_{y \in \mathbb{R}^{n+n}} \alpha(y) + y_\lambda^\top x - \rho \|x - y_w\|.$$

Below we illustrate how to compute the convex-hull

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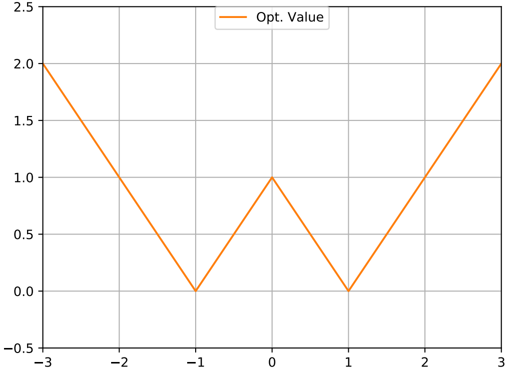

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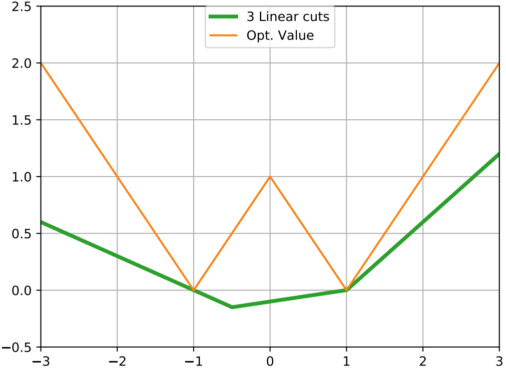

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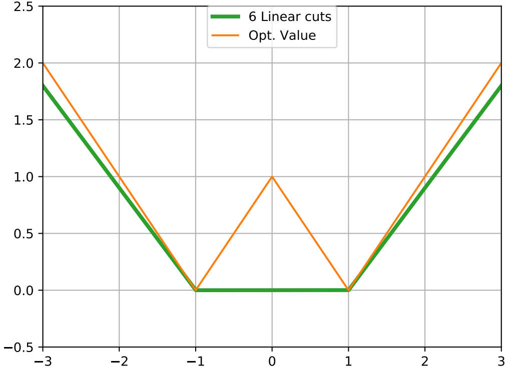

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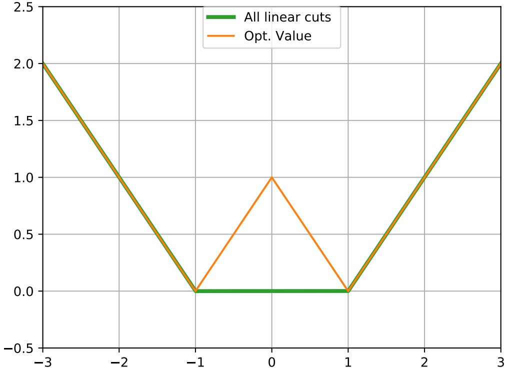

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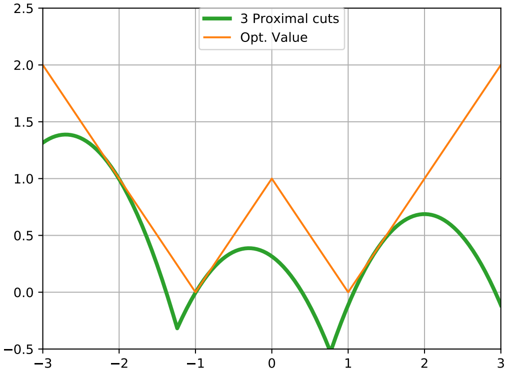

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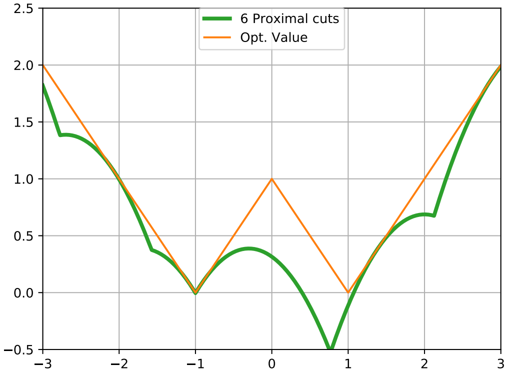

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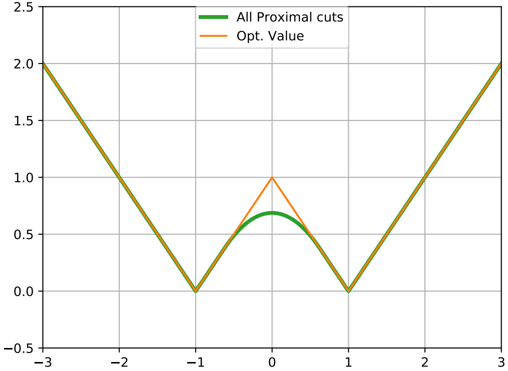

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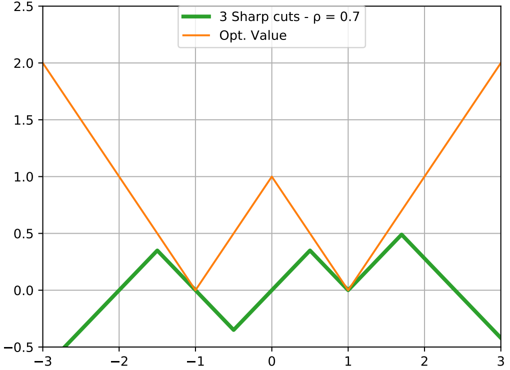

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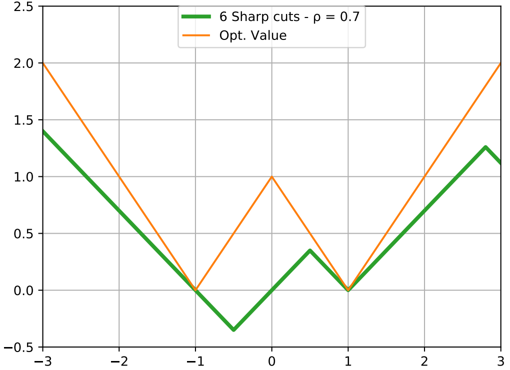

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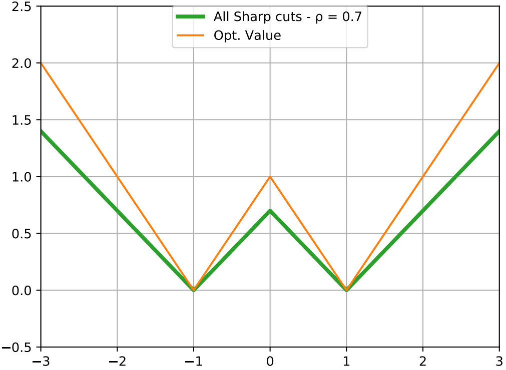

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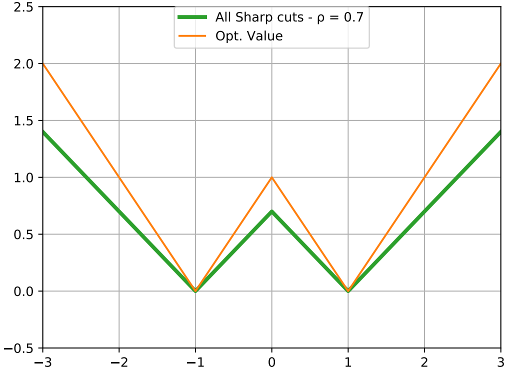

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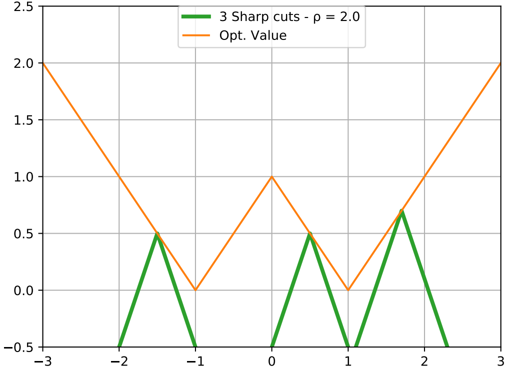

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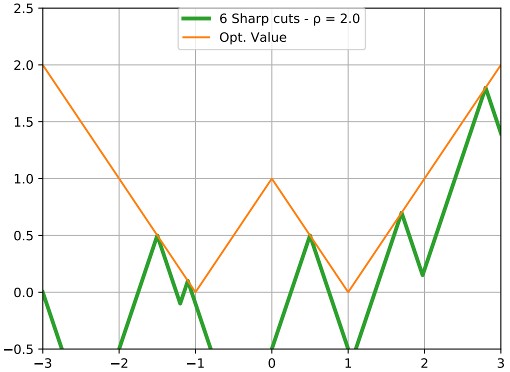

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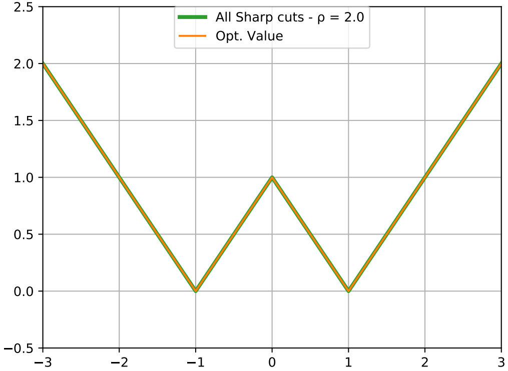

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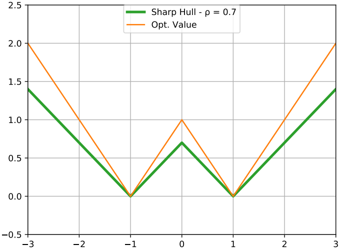
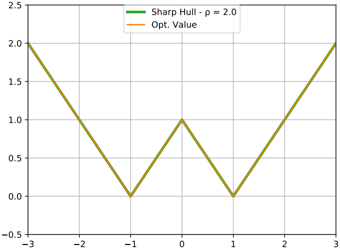


(a) $\rho < L$



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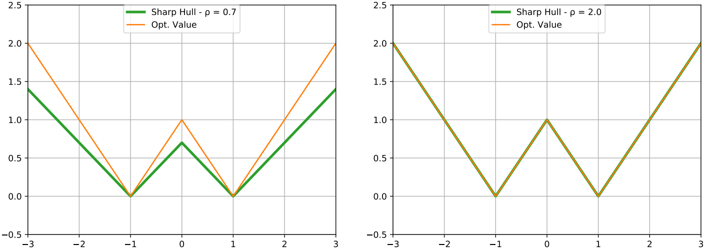

(a) $\rho < L$ (b) $\rho \geqslant L$

Question: how to use these ideas in practice?

Stochastic Lipschitz Dynamic Programming (SLDP)

In terms of algorithm, the SLDP is pretty similar to the SDDP method, but instead of computing linear Benders cuts in the backward step we compute nonlinear sharp cuts (Augmented Lagrangian cuts):

$$\overline{\mathfrak{Q}}_t^k(x) = \max\left\{\overline{\mathfrak{Q}}_t^{k-1}(x),\ \overline{\alpha}_t^k(y^k) + y_\lambda^{k\top} x - \rho_t \|x - y_w^k\|\right\},$$

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How can we use nonlinear cuts in a "tractable" way?
Answer: The reverse norms $-\|\cdot\|_1$ and $-\|\cdot\|_\infty$ are MILP representable on a compact polyhedron.

Actually, the reverse norms $-\|\cdot\|_1$ and $-\|\cdot\|_\infty$ are the only MILP representable reverse ℓ_p norms for dimension $n \geqslant 2$ [Lubin-2017].

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Let's build our intuition for the one-dimensional case:

$$\left\{(x,\gamma) \;\middle|\; \begin{array}{l} \gamma \geqslant \quad |x|, \\ x \in [-a, a] \end{array} \right\} =$$

$$\left\{(x,\gamma) \;\middle|\; \begin{array}{ll} \gamma \geqslant \quad (x^+ + x^-), & x = x^+ - x^-, \\ 0 \leqslant x^+ \leqslant \quad a, & 0 \leqslant x^- \leqslant \quad a, \\ x^+, x^- \geqslant 0, & \end{array} \right\}.$$

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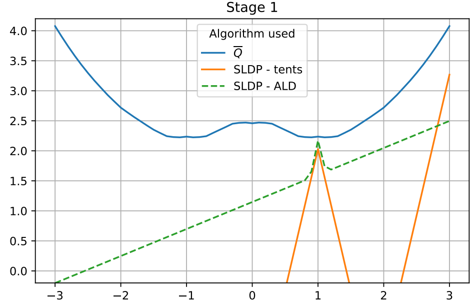
We can use MILP solvers to compute the forward step of the SLDP method.

The convergence of the SLDP algorithm is also similar to the convex case, and it is based on the convergence lemma:

$$\lim_{k \in \mathcal{K}} \overline{\mathfrak{Q}}_t^k(x_t^k) = \overline{Q}_t(x_t^*),$$

where $\{x_t^k\}_{k \in \mathcal{K}}$ is a convergent subsequence of policy states obtained in the forward step at stage t.



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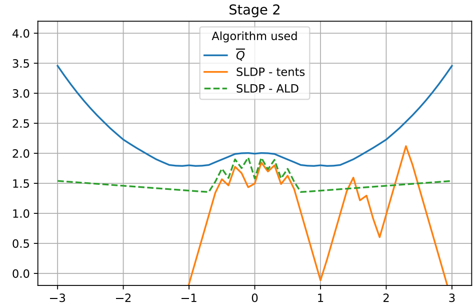

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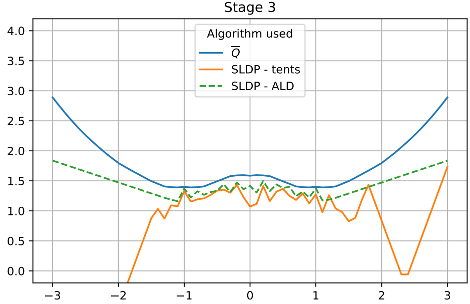

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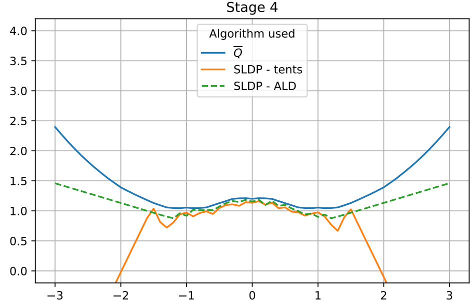

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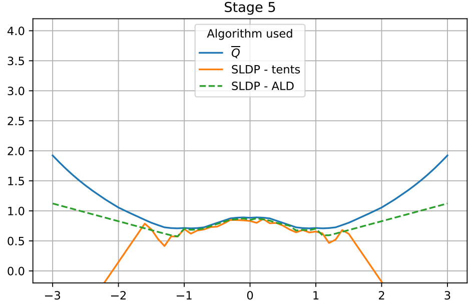

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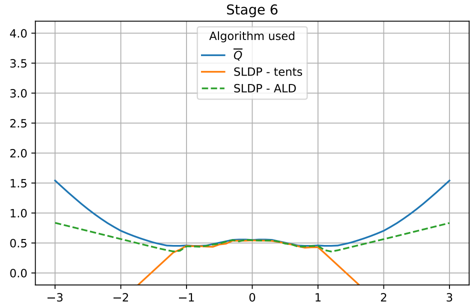

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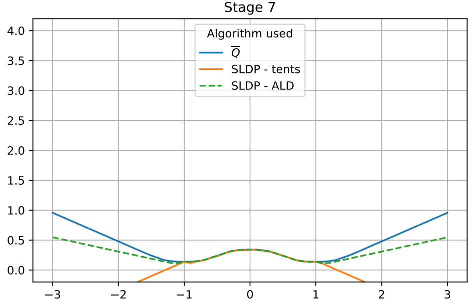

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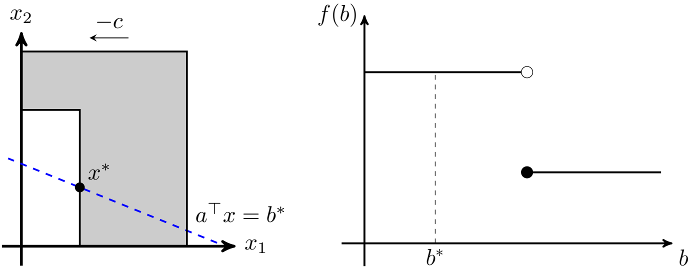
	SB	SDDiP 0.01	SLDP tents	SLDP ALD
LB	1.167	2.370	3.073	3.085
UB	3.453	3.490	3.320	3.313
time (s)	12	3317	558	605

Table: Results for an 8-stage non-convex problem

A proof or a counter-example for the convergence of the SLDP algorithm in the general MILP setting where the Complete Continuous Recourse condition do not hold.



(a) Feasible set

(b) MIP Value function

Feizollahi-2017 proves strong duality for the Sharp-family in the general MILP case.

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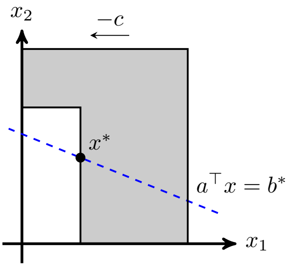

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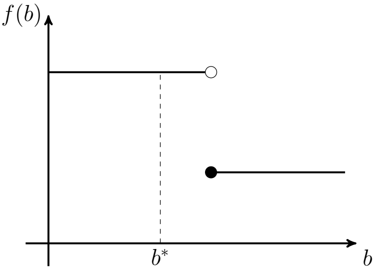

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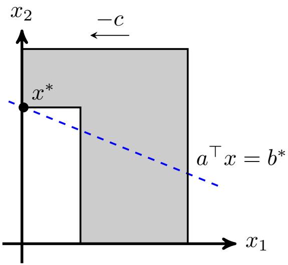

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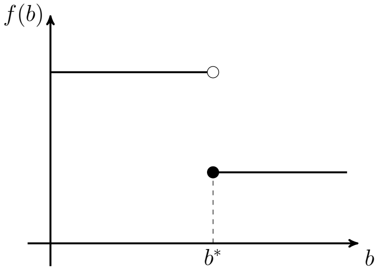

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Future work

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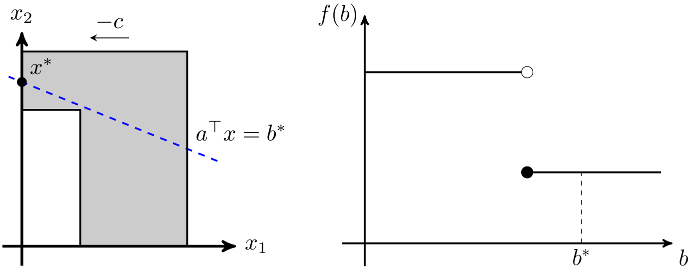

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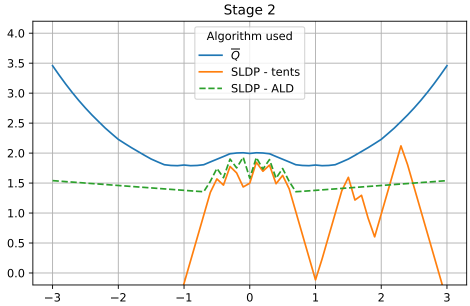
Future work

A better estimate for the penalty constant ρ of the ALD cut, since smaller values of ρ induce nonlinear cuts that fill the non-convex region using less iterations, so the lower bound increases more quickly.



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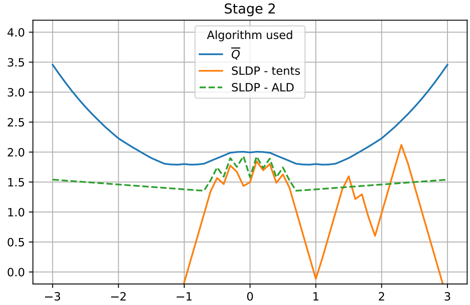

Thank you!