Marginal cost smoothing

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1 Introduction

This work emerged from a concern about volatility of the marginal costs officially published by the Brazilian operation planning model. In this document we discuss the definition of marginal cost and its volatility, the usual method for computing marginal costs, a method to assess the volatility and some proposals on how the intrinsic volatility can be regulated. This activity is part of the ONS-Gatech technical agreement.

This report is organized as follows. Section 2 describes the concept of marginal cost in the context of power system planning. In particular, it brings to the fore the fact that the total operation cost is a non-differentiable function for all demand values. The slightly more involved case of the dynamic setting is discussed and two alternate methods to evaluate the marginal costs are described and illustrated. The concept of volatility is briefly discussed in Section 3 and illustrated with the aid of two traditional time series model, AR and ARCH. The evaluation of the volatility in the context of the SDDP algorithm is described. With focus on the effects that the SDDP algorithm could have in volatility, Section 4 introduces a methodology based on quadratic regularization to smooth the marginal cost aiming at reducing this effect. Two approaches are considered: one focusing on the objective function and the other on the cost-to-go function. A case study illustrating the use of the marginal cost smoothing methodology applied to the objective function is presented in Section 5. Finally, Section 6 summarizes the main conclusions and points to some further studies.

2 Marginal cost

In economics, marginal cost is the *rate of change* (derivative) of the total cost with respect to the quantity produced, [Stoft, 2002, page 66]. In the context of hydrothermal power systems, total cost is taken as the total fuel cost of thermal plants plus penalties for constraint violations over the whole planning horizon. The produced quantity is electrical energy. Therefore, increasing or reducing energy demand by one unit would result in a different total cost. The difference in total cost can be approximated by the marginal cost multiplied by the difference in demand, if demand variation is small enough.

Most of the time the notion of marginal cost is well defined, but there are some cases where it is not. The marginal cost is not well defined for a given demand if the total cost function is non-differentiable at that demand, e.g., when the cost to produce an extra unit is distinctly greater than the savings from producing one unit less, see figure 1.

From left-side of Figure 1, taken from Stoft [2002], we note a total cost curve which has a *kink* at 10.000 MWh. This kink indicates that the cheapest thermal unit achieves its maximum capacity at 10.000 MWh. Then, a more expensive thermal unit must be committed to meet an additional load requirement. From right-side of figure 1, we observe a marginal cost curve which is piecewise constant and has a *jump* at 10.000 MWh. When the demand is less than 10.000 MWh the marginal cost is \$20/MWh and if the demand is greater than 10.000 MWh the marginal cost is \$40/MWh. However, if the demand is 10.000 MWh the marginal cost is not well defined, since the cost to produce an extra unit is \$40/MWh and the saving from producing one less is \$20/MWh, i.e., the total cost function is non-differentiable at a demand of 10.000 MWh. This is also visible by the kink in the total cost curve at 10.000 MWh.

Marginal cost definition can be extended to non-differentiable points using the notion of right-hand and left-hand marginal cost. The right-hand marginal cost is defined as the cost to produce an extra unit (right derivative) and the left-hand marginal cost is defined as the saving from producing one less unit (left derivative). Hence, when the typical marginal cost is not well defined, we still may have a notion of marginal cost if the demand is increasing or decreasing.

Fortunately, the points where the typical notion of marginal cost is ill-defined are quite rare (Lebesgue measure zero). So, it is very unlikely that any observed demand have this problem and for all practical purpose the definition of marginal cost as a derivative with respect to demand is enough.

2.1 Deterministic case

Let us start with the following mathematical setting. Consider the optimization problem

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{Min}} & f(x) \\ \text{s.t.} & Ax + b \leq 0, \end{array} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, A is an $m \times n$ matrix and b is an $m \times 1$ vector. We denote by $\vartheta(b)$ the optimal value of problem (1) considered as a function of $b \in \mathbb{R}^m$.

We are interested in sensitivity of the optimal value $\vartheta(b)$ of (1) with respect to small changes of the right hand side vector b as illustrated in Figure 1 for the deterministic planning problem, where b plays the role of demand. If $\vartheta(\cdot)$ is differentiable at a point $b \in \mathbb{R}^m$, then for small values of $h = (h_1, ..., h_m)$ we can approximate the difference $\vartheta(b + h) - \vartheta(b)$ by

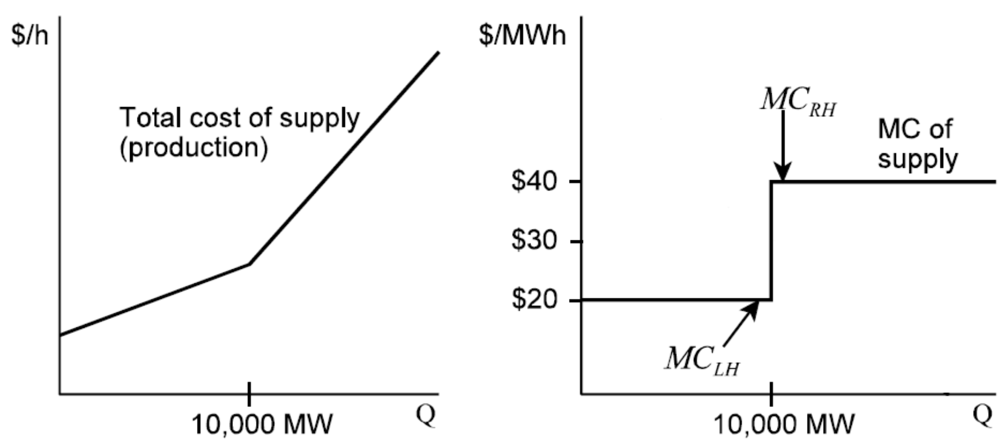

Figure 1: Total and marginal cost graphics

$h^{\mathsf{T}}\nabla\vartheta(b)$, where $\nabla\vartheta(b)$ is the $m \times 1$ gradient vector whose components are partial derivatives $\partial\vartheta(b)/\partial b_i$ and $h^{\mathsf{T}}\nabla\vartheta(b) = \sum_{i=1}^{m} h_i \frac{\partial\vartheta(b)}{\partial b_i}$. More accurately this can be written as

$$\vartheta(b+h) - \vartheta(b) = h^{\mathsf{T}}\nabla\vartheta(b) + o(\|b' - b\|). \tag{2}$$

The remainder term $o(\|b'-b\|)$ is small compared with $\|b'-b\|$, i.e., $\frac{o(\|b'-b\|)}{\|b'-b\|}$ tends to 0 as $b' \to b$.

Let us discuss some basic properties of function $\vartheta(\cdot)$. Since the objective function $f(\cdot)$ is convex, it follows that the optimal value function $\vartheta(\cdot)$ is convex. Indeed consider the following extended real valued function

$$\psi(x,b) = \begin{cases} f(x), & \text{if } Ax + b \leq 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

Then for given b, problem (1) is equivalent to minimization of $\psi(x,b)$ over $x \in \mathbb{R}^n$, and

$$\vartheta(b) = \inf_{x\in\mathbb{R}^n} \psi(x,b).$$

Since $f(x)$ is convex it follows that function $\psi(x,b)$ is convex jointly in x and b. Consequently the min-function $\vartheta(\cdot)$ is convex.

It is said that problem (1) is *polyhedral* if function $f(\cdot)$ is piecewise linear, i.e., can be represented as maximum of a finite family of affine functions

$$f(x) = \max\{\alpha_i^{\mathsf{T}} x + \beta_i : i = 1, ..., k\}$$

for some $\alpha_i \in \mathbb{R}^n$, $\beta_i \in \mathbb{R}$, $i = 1, ..., k$. In that case function $\vartheta(\cdot)$ is also piecewise linear.

The (Lagrangian) dual of problem (1) is the problem

$$\operatorname*{Max}_{\lambda\in\mathbb{R}^m_+} \left\{ \phi(\lambda) = \inf_{x\in\mathbb{R}^n} L(x,\lambda)) \right\}, \tag{3}$$

where $\mathbb{R}^m_+ = \{y \in \mathbb{R}^m : y \geq 0\}$ and

$$L(x,\lambda) = f(x) + \lambda^{\mathsf{T}}(Ax + b) \tag{4}$$

is the respective Lagrangian. Under mild regularity conditions there is no duality gap between problems (1) and (3), i.e., their optimal values are equal to each other. In particular if problem (1) is polyhedral, then it can be written as a linear programming problem and the no duality gap property holds automatically unless both the primal and dual problems are infeasible.

Recall that $g \in \mathbb{R}^n$ is said to be a subgradient of convex function $\vartheta(\cdot)$ at point $b \in \mathbb{R}^m$ if

$$\vartheta(b') \geq \vartheta(b) + g^{\mathsf{T}}(b' - b)$$

for all $b' \in \mathbb{R}^m$. The set of all subgradients of $\vartheta(\cdot)$ at b is called the subdifferential and denoted $\partial\vartheta(b)$. Convex function $\vartheta(\cdot)$ is differentiable at b iff its subdifferential $\partial\vartheta(b) = \{g\}$ is a singleton, in which case its gradient $\nabla\vartheta(b) = g$.

Consider an optimal solution $\bar{x}$ of the primal problem (1). The corresponding first order optimality conditions are: there is a vector $\lambda \in \mathbb{R}^m$ of Lagrange multipliers such that

$$0 \in \partial_x L(\bar{x}, \lambda), \tag{5}$$
$$\lambda \geq 0, \ A\bar{x} + b \leq 0, \tag{6}$$
$$\lambda^{\mathsf{T}}(A\bar{x} + b) = 0, \tag{7}$$

where $\partial_x L(\bar{x}, \lambda) = \partial f(\bar{x}) + A^{\mathsf{T}}\lambda$. In particular, if $f(\cdot)$ is differentiable at $\bar{x}$, then condition (5) takes the form

$$\nabla f(\bar{x}) + A^{\mathsf{T}}\lambda = 0. \tag{8}$$

The set of all Lagrange multipliers, denoted Λ, coincides with the set of optimal solutions of the dual problem (3) and is the same for any optimal solution of the primal problem. Note that the set Λ is associated with considered values of matrix A and vector b. Here we deal with fixed matrix A while vector b could be a subject to small changes. In order to emphasize dependence of Lagrange multipliers on vector b we sometimes write $\Lambda = \Lambda(b)$ as a function of b.

Provided that $\vartheta(b)$ is finite, the subdifferential $\partial\vartheta(b)$ is given by the set of optimal solutions of the respective dual problem (3). That is let $\Lambda = \Lambda(b)$ be the set of Lagrange multipliers satisfying conditions (5)–(7), then

$$\partial\vartheta(b) = \Lambda. \tag{9}$$

It follows that $\vartheta(\cdot)$ is differentiable at b iff problem (3) has unique optimal solution (unique Lagrange multiplier) $\bar{\lambda}$, in which case the respective gradient $\nabla\vartheta(b) = \bar{\lambda}$.

Rademacher's theorem states that if U is an open subset of $\mathbb{R}^m$ and $\phi : U \to \mathbb{R}$ is a Lipschitz continuous function, then ϕ is differentiable almost everywhere in U; that is, the points in U at which ϕ is not differentiable form a set of Lebesgue measure zero. In particular this can be applied to the convex function $\vartheta(\cdot)$ to conclude that for almost every b the set $\Lambda = \Lambda(b)$ is a singleton, i.e., primal problem (1) possesses unique Lagrange multiplier vector. An implication of this result is that if we pick up vector b at "random", then the corresponding problem (1) possesses unique Lagrange multipliers vector and $\vartheta(\cdot)$ is differentiable at b. Yet this should be taken carefully since if the considered value of b is close to a point where $\vartheta(\cdot)$ is not differentiable, the approximation $h^{\mathsf{T}}\nabla\vartheta(b)$ may be not accurate.

2.2 Dynamic setting

In the context of hydrothermal power systems, total cost is taken as the total fuel cost of thermal plants plus penalties for constraint violations over the whole planning horizon. The produced quantity is electrical energy and therefore increasing or reducing energy demand by one unit would result in a different total cost. The difference in total cost can be approximated by the marginal cost multiplied by the difference in demand, if demand variation is small enough.

In presence of uncertainty, we must define what we mean by marginal cost. Since the inflows are uncertain and our decisions depend on past inflows, the total cost over the planning horizon is also uncertain (random variable). Let us discuss now dynamic problems starting with risk neutral case. Consider the following T-stage risk neutral stochastic program (written in the nested form)

$$\underset{\substack{A_1x_1=b_1\\ x_1\geq 0}}{\text{Min}}\; c_1^{\mathsf T}x_1+\mathbb{E}\left[\underset{\substack{B_2x_1+A_2x_2=b_2\\ x_2\geq 0}}{\min} c_2^{\mathsf T}x_2+\cdots+\mathbb{E}\left[\underset{\substack{B_Tx_{T-1}+A_Tx_T=b_T\\ x_T\geq 0}}{\min} c_T^{\mathsf T}x_T\right]\right]. \tag{10}$$

In the considered case there are two types of balance equations $B_tx_{t-1}+A_tx_t = b_t$. Namely, energy conservation equations

$$SE_{t,n} = SE_{t-1,n} + CE_{t,n} - GH_{t,n} - SP_{t,n}, \tag{11}$$

relating the stored energy (SE) at the stage t to the stored energy at stage $t-1$ plus controllable energy inflow (CE) minus total hydro generated energy (GH) and losses (SP) due to spillage, evaporation, etc. And the energy balance equations

$$GH_{t,n} + \sum_{j\in NT_n} GT_{t,j} + NF_{t,n} = L_{t,n} \tag{12}$$

for satisfying load (demand) (L) at stage t, where (GT) denotes the thermal generation.

Constraints $B_tx_{t-1}+A_tx_t = b_t$ are obtained writing

$$x_t = (SE, GH, GT, SP, NF)_t^{\mathsf T},\;\; b_t = (CE, L)_t^{\mathsf T},\;\; c_t = (0,0,CT,0,0)_t^{\mathsf T}, \tag{13}$$

$$A_t = \begin{pmatrix} I & I & 0 & I & 0\\ 0 & I & \Delta & 0 & I\end{pmatrix},\;\; B_t = \begin{pmatrix} -I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}, \tag{14}$$

where $\Delta = \{\delta_{n,j} = 1 \text{ for all } j \in \mathrm{NT}_n \text{ and zero else}\}$, I and 0 are identity and null matrices, respectively, of appropriate dimensions and the components of CT are the unit operation cost of each thermal plant and penalty for failure in load supply. Note that hydroelectric generation costs are assumed to be zero. Physical constraints on variables like limits on the capacity of the equivalent reservoir, hydro and thermal generation, transmission capacity and so on are taken into account with constraints on x_t.

The inflows process CE_t is random while the load process L_t is supposed to be known (deterministic). In this model only the right hand sides b_t are random. In case the inflows process CE_t is stagewise independent, equations (13)–(14) represent constraints of the corresponding stochastic program. However, CE_t is modeled as an autoregressive time series process. In that modeling of across time dependence, variables CE_t become state variables, i.e. are components of vectors x_t, and the underline random process becomes the error process of the autoregressive model which is assumed to be stagewise independent. Note that unlike inflows equations (11), the balance equations (12) for the demand at stage t do not involve state variables at stage $t-1$.

The corresponding dynamic equations can be written going backwards in time. Because of stagewise independence of the random process $b_1, ..., b_T$, the cost-to-go (value) function at stage $t = T, ..., 2$ is

$$Q_t(x_{t-1}, b_t) = \inf_{\substack{B_t x_{t-1} + A_t x_t = b_t \\ x_t \geq 0}} c_t^{\mathsf{T}} x_t + \mathcal{Q}_{t+1}(x_t), \tag{15}$$

where[1]

$$\mathcal{Q}_{t+1}(x_t) = \mathbb{E}[Q_{t+1}(x_t, b_{t+1})] \tag{16}$$

and with the term $Q_{T+1}(\cdot)$, at stage $t = T$, omitted. At first stage the problem

$$\operatorname*{Min}_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^{\mathsf{T}} x_1 + \mathcal{Q}_2(x_1) \tag{17}$$

should be solved with $\mathcal{Q}_2(x_1) = \mathbb{E}[Q_2(x_1, b_2)]$.

The cost-to-go functions define an optimal policy $\bar{x}_1, ..., \bar{x}_T$ for problem (10), with $\bar{x}_1$ being an optimal solution of the first stage problem (17) and $\bar{x}_t$, $t = 2, ..., T$, being a minimizer of the right hand side of (15), that is

$$\bar{x}_t \in \operatorname*{arg\,min}_{\substack{B_t \bar{x}_{t-1} + A_t x_t = b_t \\ x_t \geq 0}} c_t^{\mathsf{T}} x_t + \mathcal{Q}_{t+1}(x_t). \tag{18}$$

Note that for $t = 2, ..., T$, the minimizer $\bar{x}_t = \bar{x}_t(\bar{x}_{t-1}, b_t)$ is a function of $\bar{x}_{t-1}$ and b_t. Note also that the optimal value of the first stage problem (17) is equal to the optimal value of problem (10) and represents the optimal expected *total* cost $\mathbb{E}\left[\sum_{t=1}^T c_t^{\mathsf{T}} \bar{x}_t\right]$ of the considered T-stage problem.

Consider the second stage problem

$$\operatorname*{Min}_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_1 \geq 0}} c_2^{\mathsf{T}} x_2 + \mathcal{Q}_3(x_2), \tag{19}$$

where $\mathcal{Q}_3(x_2) = \mathbb{E}[Q_3(x_2, b_3)]$. The optimal value of problem (19) represents the optimal expected total cost $\mathbb{E}\left[\sum_{t=2}^T c_t^{\mathsf{T}} \bar{x}_t\right]$ conditional on $x_1 = \bar{x}_1$ and b_2. That is, this optimal value is given by the conditional expectation and as such is a function of b_2 (and x_1). Suppose that the first stage decision vector x_1 is given (fixed). We can view x_1 as representing the initial conditions. Now problem (19) depends on realization of random vector b_2, and because of the energy balance equations (12) at $t = 2$, it also depends on demand (load) vector $d = L_2$. Let us denote by $\vartheta(b_2, d)$ the optimal value of problem (19). Note that change of the demand at the second stage can produce changes in the cost-to-go function $\mathcal{Q}_2$, but it will not effect

[1] The expectation is taken with respect to the distribution of b_{t+1}.

cost-to-go functions $\mathcal{Q}_t$ for $t = 3, ..., T$. This is because the cost-to-go functions are computed backwards in time.

A natural question is how $\vartheta(b_2, d)$ is related to $Q_2(x_1, b_2)$. By the definition of the function $Q_2(x_1, b_2)$ (see equation (15)), for the given value of the demand $d = L_2$ we have that $\vartheta(b_2, d) = Q_2(x_1, b_2)$. We are going to consider now sensitivity of the optimal value $\vartheta(b_2, d)$ for small changes of the demand levels at the second stage.

2.2.1 The usual approach

The function $\vartheta(b_2, d)$ is convex. The subdifferential of $\vartheta(b_2, d)$, with respect to d, is given by the set of Lagrange multipliers associated with the energy balance equations (at stage $t = 2$) with the minus sign. That is, let $\bar{\lambda}(b_2, d)$ be a respective vector of Lagrange multipliers. Assuming that this vector of Lagrange multipliers is unique, we have that the gradient of $\vartheta(b_2, d)$ with respect[2] to d is

$$\nabla_d \vartheta(b_2, d) = -\bar{\lambda}(b_2, d). \tag{20}$$

We can give the following interpretation of the Lagrange multipliers vector $\bar{\lambda}(b_2, d)$.

> Provided that the Lagrange multipliers vector $\bar{\lambda}(b_2, d)$ is unique, we have that $-\bar{\lambda}(b_2, d)$ represents the gradient of changes of the expected *total* cost $\mathbb{E}\left[\sum_{t=2}^{T} c_t^{\mathsf{T}} \bar{x}_t\right]$, conditional on $x_1 = \bar{x}_1$ and b_2, with respect to small variations of the demand vector at the second stage of the process.

By computing the expectation $\mathbb{E}[-\bar{\lambda}(b_2, d)]$ we evaluate the gradient of the respective expected value of the *total* cost.

In the above framework consider now the following risk averse case. For risk measure

$$\rho_t(\cdot) := (1 - \lambda)\mathbb{E}(\cdot) + \lambda \mathsf{AV@R}_\alpha(\cdot), \tag{21}$$

consider problem

$$\underset{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}}{\text{Min}} c_1^{\mathsf{T}} x_1 + \rho_2 \left[\min_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} c_2^{\mathsf{T}} x_2 + \cdots + \rho_T \left[\min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^{\mathsf{T}} x_T \right] \right]. \tag{22}$$

Of course for $\lambda = 0$, problem (22) coincides with the risk neutral problem (10).

The corresponding dynamic equations can be written as follows. The cost-to-go (value) function at stage $t = T, ..., 2$ is

$$Q_t(x_{t-1}, b_t) := \inf_{\substack{B_t x_{t-1} + A_t x_t = b_t \\ x_t \geq 0}} c_t^{\mathsf{T}} x_t + \mathcal{Q}_{t+1}(x_t), \tag{23}$$

[2]Note that here we take Lagrange multiplier in (20) with minus sign since feasibility equations $B_t x_{t-1} + A_t x_t - b_t = 0$ are written with minus sign in b_t.

where

$$\mathcal{Q}_{t+1}(x_t) = \rho_{t+1}[Q_{t+1}(x_t, b_{t+1})] \tag{24}$$

and with the term $\rho_{T+1}(\cdot)$, at stage $t = T$, omitted. At first stage the problem

$$\operatorname*{Min}_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\mathsf{T} x_1 + \mathcal{Q}_2(x_1) \tag{25}$$

should be solved, where $\mathcal{Q}_2(x_1) = \rho_2[Q_2(x_1, b_2)]$.

Consider the second stage problem

$$\operatorname*{Min}_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_1 \geq 0}} c_2^\mathsf{T} x_2 + \mathcal{Q}_3(x_2), \tag{26}$$

where $\mathcal{Q}_3(x_2) = \rho_3[Q_3(x_2, b_3)]$. The difference between this risk averse formulation and the above risk neutral setting is that here the optimal value of the first stage problem (25) represents the optimal value of problem (22). Similarly the optimal value of problem (26) is equal to the optimal value of the risk averse problem

$$\operatorname*{Min}_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} c_2^\mathsf{T} x_2 + \rho_3 \left[\min_{\substack{B_3 x_2 + A_3 x_3 = b_3 \\ x_3 \geq 0}} c_3^\mathsf{T} x_3 + \cdots + \rho_T \left[\min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^\mathsf{T} x_T \right] \right], \tag{27}$$

conditional on x_1 and b_2.

Consider the respective vector of Lagrange multipliers $\bar{\lambda}(b_2, d)$ associated with the energy balance equation. This vector is defined in the same way as in the risk neutral case with the cost-to-go function $\mathcal{Q}_3(\cdot)$ derived by the dynamic equations (23). Here it has the following interpretation.

> Provided that the Lagrange multipliers vector $\bar{\lambda}(b_2, d)$ is unique, we have that $-\bar{\lambda}(b_2, d)$ represents the gradient of changes of the optimal value of the risk averse problem (27), conditional on $x_1 = \bar{x}_1$ and b_2, with respect to small variations of the demand vector at the second stage of the process.

Note that the optimal value of risk averse problem (27) involves penalties for extreme costs imposed by the AV@R part of the risk measures. It does not make much sense here to average $\bar{\lambda}(b_2, d)$, i.e., to compute $\mathbb{E}[\bar{\lambda}(b_2, d)]$, over the distribution of b_2. It does not make sense not only due to interpretation of the economic meaning of the penalties, but also because of the nested nature of the mean-AV@R objective function.

3 Volatility

Volatility is a synonym for unpredictability. A judicious description of unpredictability requires the definition of prediction. Prediction is a guess about a *future* random quantity.

The typical guess is the (conditional) expectation of the quantity which is a predictor with the property of minimum mean squared error:

$$\widehat{X}_t = \mathbb{E}[X_t \mid X_{t-1}, X_{t-2}, \dots].$$

On these terms, volatility can be seen as a *dispersion* of all possible future outcomes around a prediction conditional on the available information $X_{[t-1]} := (X_{t-1}, X_{t-2}, \dots)$:

$$\text{Volatility}_t := \text{dispersion}[(X_t - \widehat{X}_t) \mid X_{[t-1]}]. \tag{28}$$

One typical measure of dispersion that is also related to the conditional expectation is the conditional standard deviation. By definition, (conditional) standard deviation is the square root of the (conditional) mean squared error between all possible random outcomes and the prediction given by the expected value:

$$\text{Volatility}_t = \mathbb{E}[(X_t - \widehat{X}_t)^2 \mid X_{[t-1]}]^{1/2} = \text{Var}[X_t \mid X_{[t-1]}]^{1/2}. \tag{29}$$

In these terms, a random quantity is more volatile if the conditional mean squared error of prediction, that is the conditional standard deviation, is bigger.

3.1 Volatility of AR(p) and ARCH(p) models

In order to illustrate the volatility concept, we present below two time-series models. The first one is an autoregressive AR(p) model:

$$X_t = \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \sigma \epsilon_t, \tag{30}$$

where ϵ_t is an iid process with ϵ_t having standard normal distribution, $\epsilon_t \sim \mathcal{N}(0, 1)$, and $\sigma > 0$ is a fixed constant. By definition, the volatility of the random outcome X_t at time $t-1$ is the conditional standard deviation of X_t given the available information $X_{[t-1]}$.

$$\begin{aligned}
\text{Var}[X_t \mid X_{[t-1]}]^{1/2} &= \text{Var}[\phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \sigma \epsilon_t \mid X_{[t-1]}]^{1/2} \\
&= \text{Var}[\sigma \epsilon_t \mid X_{[t-1]}]^{1/2} \\
&= \sigma,
\end{aligned}$$

in other words, the volatility of an AR(p) model is constant and equal to σ. It is instructive to emphasize that the standard deviation (*without* conditional on past observation) of an AR(p) model is not equal to σ. Consider, for instance, an AR(1) model:

$$X_t = \phi_0 + \phi_1 X_{t-1} + \sigma \epsilon_t,$$

with $|\phi_1| < 1$. For this model, the variance of X_t has the following expression:

$$\begin{aligned}
\text{Var}[X_t] &= \text{Var}[\phi_0 + \phi_1 X_{t-1} + \sigma \epsilon_t] \\
&= \phi_1^2 \,\text{Var}[X_{t-1}] + \sigma^2 \,\text{Var}[\epsilon_t] + 2\phi_1 \sigma \,\text{Cov}[X_{t-1}, \epsilon_t].
\end{aligned} \tag{31}$$

Under week stationarity condition (guaranteed by $|\phi_1| < 1$), we have that

$$\mathrm{Cov}[X_{t-1}, \epsilon_t] = 0, \quad \mathrm{Var}[X_t] = \mathrm{Var}[X_{t-1}].$$

We conclude from (31) the standard deviation formula:

$$\mathrm{Var}[X_t]^{1/2} = \frac{\sigma}{\sqrt{1-\phi_1^2}},$$

which is not equal to $\mathrm{Var}[X_t \mid X_{[t-1]}]^{1/2} = \sigma$.

The second model is the ARCH(p) (Autoregressive Conditional Heteroscedasticity) process (of order p). In this case, the random outcome X_t is defined by

$$X_t = \sigma_t Z_t,$$

where Z_t is an iid process with Z_t having standard normal distribution, $Z_t \sim \mathcal{N}(0,1)$, and σ_t is a function of $X_{t-1}, \ldots, X_{t-p}$ of the form

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2. \tag{32}$$

For this model, conditional expectation is zero:

$$\mathbb{E}[X_t \mid X_{[t-1]}] = \mathbb{E}[\sigma_t Z_t \mid X_{[t-1]}] = \sigma_t \mathbb{E}[Z_t \mid X_{[t-1]}] = 0,$$

and conditional standard deviation is σ_t:

$$\mathrm{Var}[X_t \mid X_{[t-1]}] = \mathbb{E}[X_t^2 \mid X_{[t-1]}] = \sigma_t^2 \mathbb{E}[Z_t^2 \mid X_{[t-1]}] = \sigma_t^2.$$

Thus, volatility of an ARCH model is not constant and depends on the last p observations:

$$\mathrm{Var}[X_t \mid X_{[t-1]}]^{1/2} = \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2}.$$

In summary, volatility and time-series model are closely related concepts, since definition of volatility requires the notion of conditional distribution, which in turn depends on the model specification. An usual volatility measure is the conditional standard deviation. We emphasize that the standard deviation itself and the conditional standard deviation are two different concepts, as illustrated in the AR(1) example.

3.2 Volatility of marginal cost

We have two main approaches to assess the marginal cost volatility:

1. Fit a time-series model on the historical data of marginal cost, an ARCH model for instance, and estimate the conditional standard deviation;

2. Fit a model for the initial condition variables, simulate several input for the stochastic optimization model and estimate the conditional standard deviation of marginal cost by Monte Carlo;

In the Brazilian power system, marginal cost is obtained by a rolling horizon approach, i.e., on every stage the stochastic programming problem is solved from scratch with revised parameters. The typical procedure used to assess the marginal cost volatility is by fitting a model for initial condition variables (alternative 2) and simulating possible values of marginal cost. Intuitively, this seems a better approach as compared to alternative 1.

In this way, we highlight some parameters of the stochastic programming problem that could have a significant impact on the marginal cost value using the second approach. We assume, without loss of generality, that we are at time zero ($t = 0$) trying to predict marginal cost π_1 of time one ($t = 1$):

- v_1 – initial stored volume (measured) at the beginning of time 1;
- $\widehat{a}_{[1,s]}$ – deterministic inflow scenario forecast, where s is the number of considered stages. A typical value of s is 4 (weeks);
- $\tilde{a}_{[s+1,T]}$ – inflow scenarios generated by the PAR (Periodic Autoregressive model) conditioned on past observation $a_{[0]}$ and deterministic forecast $\widehat{a}_{[1,s]}$. A typical value of T is 5;
- $\mathfrak{Q}_{T+1}$ – mean-AV@R cost-to-go function computed by the Long-term planning model (problem boundary condition);
- $C_{[1,T]}$ – unit costs (deterministic) for thermal generation and deficit (load shedding), and penalties for slack variables from time 1 up to the end of horizon T.

In short, marginal cost π_1 is a function of the above mentioned parameters:

$$\pi_1 = \pi_1 \left(v_1,\, \widehat{a}_{[1,s]},\, \tilde{a}_{[s+1,T]},\, \mathfrak{Q}_{T+1},\, C_{[1,T]} \right). \tag{33}$$

Suppose we have a statistical model for evolution of the initial storage v_1, the inflow forecasts $\widehat{a}_{[1,s]}$ and the inflow scenarios $\tilde{a}_{[s+1,T]}$. For instance, we may fit a time-series model, such as the PAR model, for both $\widehat{a}_{[1,s]}$ and $\tilde{a}_{[s+1,T]}$ and we could define the initial storage v_1 as the optimal solution of the final storage from the previous rolling horizon problem. The PAR model depends on the observed inflows $a_{[0]}$ up to time zero and initial storage v_1 depends on the previous initial storage v_0, inflow forecast $\widehat{a}^{\text{Prev}}_{[1,s]}$ and inflow scenarios $\tilde{a}^{\text{Prev}}_{[s+1,T]}$.

Suppose also that we ignore eventual variability in estimation of the mean-AV@R cost-to-go function $\mathfrak{Q}_{T+1}$ and changes on the unit costs $C_{[1,T]}$. Therefore,

$$\pi_1 = \pi_1 \left(v_1,\, \widehat{a}_{[1,s]},\, \tilde{a}_{[s+1,T]} \right). \tag{34}$$

From time zero viewpoint, the *volatility of marginal cost* is the conditional standard deviation of marginal cost given the initial storage v_0 at time zero, the observed inflow $a_{[0]}$ up to time zero, the previous value of inflow forecast $\widehat{a}_{[1,s]}^{\text{Prev}}$ and inflow scenarios $\tilde{a}_{[s+1,T]}^{\text{Prev}}$:

$$\text{Var}\left[\pi_1(v_1, \widehat{a}_{[1,s]}, \tilde{a}_{[s+1,T]}) \mid (v_0, a_{[0]}, \widehat{a}_{[1,s]}^{\text{Prev}}, \tilde{a}_{[s+1,T]}^{\text{Prev}})\right]^{1/2}. \tag{35}$$

We can estimate quantity (35) by Monte Carlo, if we simulate $(v_1, \widehat{a}_{[1,s]}, \tilde{a}_{[s+1,T]})$ conditional on $(v_0, a_{[0]}, \widehat{a}_{[1,s]}^{\text{Prev}}, \tilde{a}_{[s+1,T]}^{\text{Prev}})$ and compute $\pi_1(v_1, \widehat{a}_{[1,s]}, \tilde{a}_{[s+1,T]})$.

4 Marginal cost smoothing

One problem with the current approach is that there are situations where small variations on the initial inflows result in considerable changes on marginal costs value when compared to the same variation on initial storage.

We present below a discussion about the derivative of the first stage Cost-to-go function $Q(v_1, a_0)$ with respect to the initial inflow a_0 and storage volume v_1. This may give us a clue on the marginal cost sensitivity regarding those variables. Suppose we have a risk neutral multi-stage planning model with just one thermal and one hydro plant, where the inflow model is an AR(1) process. Just to simplify notation, let $\phi_0 = 0$ and $\phi_1 = \rho$. Then,

$$\begin{aligned} Q(v_1, a_0) := \min \mathbb{E}\ [\ & c_1 g_1 + c_2 g_2 + c_3 g_3\ ] \\ \text{s.t.} \quad & v_2 = v_1 + a_1 - q_1 - s_1 \\ & a_1 = \rho a_0 + \epsilon_1 \\ & v_3 = v_2 + a_2 - q_2 - s_2 \, . \\ & a_2 = \rho a_1 + \epsilon_2 \\ & v_4 = v_3 + a_3 - q_3 - s_3 \\ & a_3 = \rho a_2 + \epsilon_3 \end{aligned} \tag{36}$$

The only uncertainty is the additive error ϵ_t, which is assumed to be stagewise independent. We restate problem (36) in a suitable form to compare derivatives. Formulation below is obtained by replacing recursively the time series expression, $a_t = \rho a_{t-1} + \epsilon_t$, in each hydro balance equation:

$$\begin{aligned} Q(v_1, a_0) = \min \mathbb{E}\ [\ & c_1 g_1 + c_2 g_2 + c_3 g_3\ ] \\ \text{s.t.} \quad & v_2 = v_1 - q_1 + (\rho a_0 + \epsilon_1) \\ & v_3 = v_1 - q_1 - q_2 + (\rho a_0 + \epsilon_1) + (\rho^2 a_0 + \rho \epsilon_1 + \epsilon_2) \\ & v_4 = v_1 - \textstyle\sum_{t=1}^{3} q_t + \sum_{t=1}^{3} (\rho^t a_0 + \sum_{j=1}^{t} \rho^{t-j} \epsilon_j) \end{aligned} \tag{37}$$

Based on (37), we define a Cost-to-go function $U(b_1, b_2, b_3)$ as a similar planning problem, but with decoupled recourse on each stage:

$$U(b_1, b_2, b_3) := \min \mathbb{E}\,[\; c_1 g_1 + c_2 g_2 + c_3 g_3 \;] \\ \text{s.t.} \quad \begin{array}{l} v_2 = b_1 - q_1 + \epsilon_1 \\ v_3 = b_2 - q_1 - q_2 + \epsilon_1 + \rho\epsilon_1 + \epsilon_2 \\ v_4 = b_3 - \sum_{t=1}^{3} q_t + \sum_{t=1}^{3}\sum_{j=1}^{t} \rho^{t-j}\epsilon_j \end{array} . \tag{38}$$

Note that the relation between both Cost-to-go functions is given by

$$Q(v_1, a_0) = U(v_1 + \rho a_0,\ v_1 + (\rho + \rho^2)a_0,\ v_1 + \textstyle\sum_{t=1}^{3} \rho^t a_0). \tag{39}$$

Assuming differentiability of U at the given condition, we may use the chain rule to conclude the relation between $\frac{\partial Q}{\partial v_1}$ and $\frac{\partial Q}{\partial a_0}$:

$$\begin{aligned} \frac{\partial Q}{\partial v_1} &= \frac{\partial U}{\partial b_1} + \frac{\partial U}{\partial b_2} + \frac{\partial U}{\partial b_3} \\ \frac{\partial Q}{\partial a_0} &= \rho\frac{\partial U}{\partial b_1} + (\rho + \rho^2)\frac{\partial U}{\partial b_2} + (\rho + \rho^2 + \rho^3)\frac{\partial U}{\partial b_3}, \end{aligned}$$

or more concisely,

$$\frac{\partial Q}{\partial a_0} = \rho\frac{\partial Q}{\partial v_1} + \rho^2\frac{\partial U}{\partial b_2} + (\rho^2 + \rho^3)\frac{\partial U}{\partial b_3}.$$

Although the relation among $\frac{\partial U}{\partial b_1}$, $\frac{\partial U}{\partial b_2}$ and $\frac{\partial U}{\partial b_3}$ is not obvious, we suspect that the last two terms $\rho^2\frac{\partial U}{\partial b_2} + (\rho^2 + \rho^3)\frac{\partial U}{\partial b_3}$ may explain the stronger marginal cost sensitivity with respect to the past inflows when compared to the initial stored volume.

This behavior is often perceived as not intuitive and methodologies have been proposed to control marginal cost variability induced by initial inflow changes. Many aspects regarding the relative importance of the inflow models have already been discussed in previous investigations. One such work that deserves mentioning is [Soares et al., 2014] approach that proposes to decouple the forward and backward steps of the SDDP algorithm. The driving idea of Soares et al. is to reduce the decisions and marginal costs variability induced by the use of PAR(p) models by using a stagewise independent inflow model.

On the other hand, the SDDP iterative algorithm convergence criterion is based on stability evaluation of some measure, say the mean-AV@R risk measure, regarding the total cost along the planning horizon. For instance, one stops the algorithm when the lower bound of the total cost measure does not change significantly along iterations. This criterion aims at the stability of the cost and does not provide any analogous provision regarding the stability of marginal costs.

One possible attempt to reduce variability is to consider a larger number of SDDP iterations. Another one is to increase the size of the scenario tree. Nevertheless, due to the high dimension of the cost-to-go function, one can not assure that any of these approaches would result in a significant reduction, Pereira [2008].

The aforementioned investigations have focused on the state variables. Instead, we investigated the possible effects on marginal cost variability due to the convergence criteria as well as the discretization used to represent the scenario tree.

In this report, we proposed two alternative approaches to deal with the possible effects of the algorithm on the solution variability.

- The first alternative addresses the possible occurrence of multiple solutions, or highly similar ones, with approximately equal total cost. This can lead to a variation of the decisions variables as well as of the corresponding marginal costs. The approach is to add a quadratic regularization term to the value function:

$$Q_t\left(x_{t-1}, \xi_t\right) = \inf_{x_t \in \mathbb{R}^{n_t}} \left\{ c_t^\intercal x_t + \mathcal{Q}_{t+1}(x_t) + \tfrac{1}{2}\varepsilon_t \|x_t - \bar{x}_t\|^2 : B_t x_{t-1} + A_t x_t = b_t,\ x_t \geq 0 \right\},$$

  which has the additional benefit of providing a unique solution to the problem.

- The second alternative addresses the stability of the marginal cost regarding small perturbations on the demand. The aim is to investigate whether regularizing the Cost-to-go function with respect to the demand could help stabilize the marginal cost.

$$Q_t^{\text{reg}}\left(x_{t-1}, \xi_t\right) = \inf_{y_{t-1} \in \mathbb{R}^{n_{t-1}}} \left\{ Q_t\left(y_{t-1}, \xi_t\right) + \frac{1}{2\gamma_t}\|y_{t-1} - x_{t-1}\|^2 \right\}.$$

In the following sections we will consider a specific notation to describe the regularization approaches. Note that the state variable x_t contains the storage volume v_t and the demand d_t, and so we can write $x_{t-1} = (v_t, d_t, \tilde{x}_{t-1})$. For brevity, we omit $\tilde{x}_{t-1}$ in the following equations. We use a particular description of the problem

$$Q_t\left(v_t, d_t\right) = \inf_{v_{t+1} \geq 0} \left\{ \text{IC}_t(v_{t+1}, v_t, d_t) + \mathcal{Q}_{t+1}(v_{t+1}, d_{t+1}) \right\}, \tag{40}$$

in which the thermal costs, hydro and load balance equation, and other box constraints on thermal, hydro, spill, and net flow variables are summarized on the immediate cost function $\text{IC}_t(v_{t+1}, v_t, d_t)$.

4.1 Objective function quadratic regularization (Tichonov)

One of the problems with the current approach is that in some cases small variations in the inflows result in considerable changes in the output marginal costs. One possible attempt to deal with this is to add a quadratic regularization term to the value (cost-to-go) functions. That is, the dynamic programming equations are modified to

$$Q_t\left(x_{t-1}, \xi_t\right) = \inf_{x_t \in \mathbb{R}^{n_t}} \left\{ c_t^\intercal x_t + \mathcal{Q}_{t+1}(x_t) + \tfrac{1}{2}\varepsilon_t \|x_t - \bar{x}_t\|^2 : B_t x_{t-1} + A_t x_t = b_t,\ x_t \geq 0 \right\}, \tag{41}$$

with $\mathcal{Q}_{t+1}\left(x_t\right) = \rho_{t+1}\left[Q_{t+1}\left(x_t, \xi_{t+1}\right)\right]$, and the additional regularization term $\frac{1}{2}\varepsilon_t\|x_t\|^2$, where $\|x_t\|^2 = x_t^\intercal x_t$ and $\varepsilon_t > 0$. Adding such regularization term may stabilize the problem.

Such approach is known as Tichonov regularization. The idea is that the optimal policy $\bar{x}_t = \bar{x}_t(x_{t-1}, \xi_t)$, $t = 2, ..., T$, is a solution of the respective minimization problem

$$\operatorname*{Min}_{x_t \in \mathbb{R}^{n_t}} \left\{ c_t^\mathsf{T} x_t + \mathcal{Q}_{t+1}(x_t) : B_t x_{t-1} + A_t x_t = b_t,\ x_t \geq 0 \right\}. \tag{42}$$

If the cost-to-go function $\mathcal{Q}_{t+1}(x_t)$ is "flat", then this optimal solution can be unstable. Adding the quadratic term $\frac{1}{2}\varepsilon_t \|x_t\|^2$ makes the objective function of problem (42) better conditioned, and hence $\bar{x}_t = \bar{x}_t(x_{t-1}, \xi_t)$ less sensitive to small changes of ξ_t.

As far as the SDDP method is concerned, this requires only to add the term

$$\nabla\left(\tfrac{1}{2}\varepsilon_t \|x_t^k\|^2\right) = \varepsilon_t x_t^k$$

to the gradient of the value function at the backward step of the algorithm at trial points x_t^k.

4.1.1 Regularization with respect to the storage

Using the particular notation (40) described earlier, one can also write equation (41) in the following way:

$$Q_t(v_t, d_t) = \inf_{v_{t+1} \geq 0} \left\{ \mathrm{IC}_t(v_{t+1}, v_t, d_t) + \mathcal{Q}_{t+1}(v_{t+1}, d_{t+1}) + \tfrac{1}{2}\varepsilon_t \|v_{t+1} - \bar{v}_{t+1}\|^2 \right\}, \tag{43}$$

which emphasizes that the quadratic regularization center is the stored volume $\bar{v}_{t+1}$.

4.2 Optimal value quadratic regularization (Moreau-Yosida)

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper, lower semi-continuous and convex function. We define as the Moreau-Yosida envelope $e_\gamma f(x)$ and proximal mapping $P_\gamma f(x)$ (set of optimal solutions) the following functions:

$$e_\gamma f(x) := \inf_w \left\{ f(w) + \frac{1}{2\gamma}\|w - x\|^2 \right\}, \tag{44}$$

$$P_\gamma f(x) := \arg\min_w \left\{ f(w) + \frac{1}{2\gamma}\|w - x\|^2 \right\}. \tag{45}$$

The regularized function $e_\gamma f(x)$ is also called the *infimal convolution* between $f(x)$ and $\frac{1}{2\gamma}\|x\|^2$. Below we summarize some properties of $e_\gamma f(x)$ and $P_\gamma f(x)$.

Consider the particular case where $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a lower semi-continuous, proper, and convex function. For this specific condition, a result from the book of [Rockafellar and Wets, 2011] regarding Moreau-Yosida envelope $e_\gamma f$ and proximal mappings $P_\gamma f$ states that:

1. the value $e_\gamma f(x)$ is finite and depends continuously on (γ, x), with

$$e_\gamma f(x) \nearrow f(x) \text{ for all } x \text{ as } \gamma \searrow 0.$$

2. The proximal mapping $P_\gamma f$ is single-valued (unique optimal solution) and continuous. In fact $P_\gamma f(x) \to P_{\bar{\gamma}} f(\bar{x})$ whenever $(\gamma, x) \to (\bar{\gamma}, \bar{x})$ with $\bar{\gamma} > 0$.

3. The envelope function $e_\gamma f$ is convex and continuously differentiable, the gradient being

$$\nabla e_\gamma f(x) = \frac{1}{\gamma}[x - P_\gamma(x)].$$

Note that the above results are valid for the hydrothermal planning problem at hand. These properties motivate the use of Moreau-Yosida regularization on Cost-to-go functions.

4.2.1 Example: absolute value function

Let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function, $f(x) = |x|$. One can prove the following formulas:

$$e_\gamma f(x) = \begin{cases} -x - \frac{\gamma}{2} & x < -\gamma \\ \frac{x^2}{2\gamma} & x \in [-\gamma, \gamma] \\ x - \frac{\gamma}{2} & x > \gamma \end{cases}$$

$$P_\gamma f(x) = \begin{cases} -x + \gamma & x < -\gamma \\ 0 & x \in [-\gamma, \gamma] \\ x - \gamma & x > \gamma \end{cases}$$

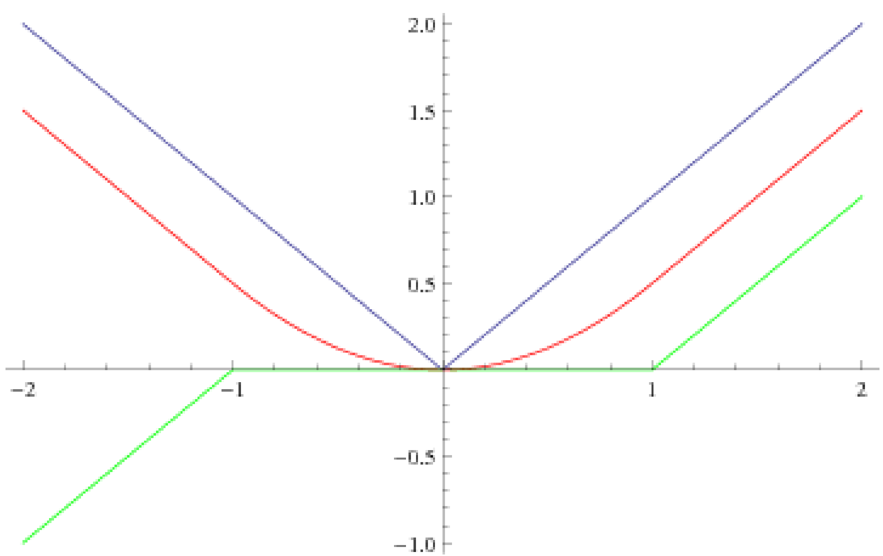

Figure 2: For $\gamma = 1$, Absolute value function $f(x) = |x|$ (blue), Moreau-Yosida envelope $e_\gamma f(x)$ (red) and proximal mapping $P_\gamma f(x)$ (green).

Figure 3 illustrates the behavior of Moreau-Yosida envolope for $\gamma \in (0, 1]$:

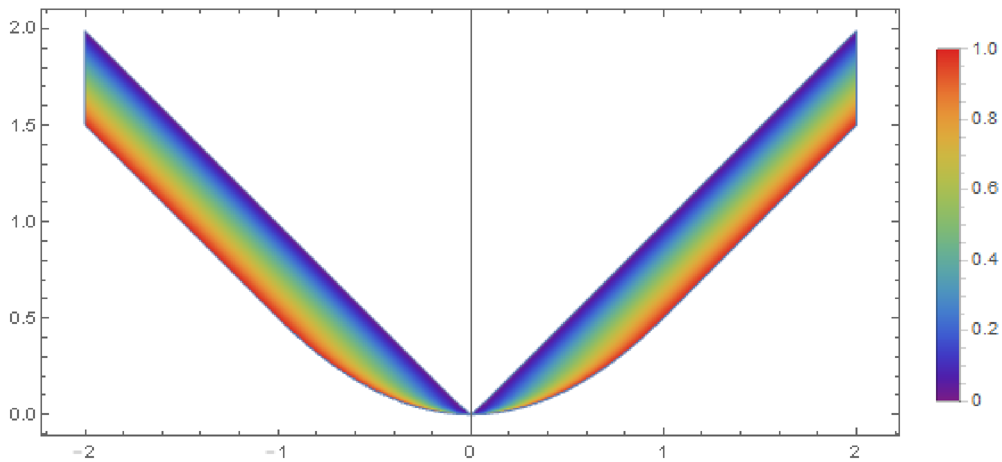

Figure 3: Absolute value Moreau-Yosida envolope $e_\gamma f(x)$ for $\gamma \in (0,1]$.

4.2.2 Cost-to-go function regularization with respect to the demand

Consider the Moreau-Yosida regularization of the Cost-to-go function with respect to the demand:

$$Q_t^{\text{reg}}(v_t, d_t) = \inf_{\tilde{d}_t \in \mathbb{R}^{m_t}} \left\{ Q_t(v_t, \tilde{d}_t) + \frac{1}{2\gamma_t} \|\tilde{d}_t - d_t\|^2 \right\}. \tag{46}$$

The following steps show how to frame this regularization as a Moreau-Yosida regularization of the Immediate cost.

$$\begin{aligned} Q_t^{\text{reg}}(v_t, d_t) &= \inf_{\tilde{d}_t \in \mathbb{R}^{m_t}} \left\{ Q_t(v_t, \tilde{d}_t) + \frac{1}{2\gamma_t} \|\tilde{d}_t - d_t\|^2 \right\} \\ &= \inf_{\substack{\tilde{d}_t \in \mathbb{R}^{m_t} \\ v_{t+1} \geq 0}} \left\{ \text{IC}_t(v_{t+1}, v_t, d_t) + \mathcal{Q}_{t+1}(v_{t+1}, d_{t+1}) + \frac{1}{2\gamma_t} \|\tilde{d}_t - d_t\|^2 \right\} \\ &= \inf_{v_{t+1} \geq 0} \left\{ \inf_{\tilde{d}_t \in \mathbb{R}^{m_t}} \left\{ \text{IC}_t(v_{t+1}, v_t, d_t) + \frac{1}{2\gamma_t} \|\tilde{d}_t - d_t\|^2 \right\} + \mathcal{Q}_{t+1}(v_{t+1}, d_{t+1}) \right\} \\ &= \inf_{v_{t+1} \geq 0} \left\{ \text{IC}_t^{\text{reg}}(v_{t+1}, v_t, d_t) + \mathcal{Q}_{t+1}(v_{t+1}, d_{t+1}) \right\}, \end{aligned}$$

Note that, as far as the regularization with respect to the demand, the above reasoning shows the equivalence between the Cost-to-go function regularization and solving the standard problem with a regularized Immediate Cost.

How to solve this problem under the SDDP algorithm:

- Regularize after the last iteration (final simulation). Use a quadratic solver;
- Regularize along the iterations, that is, similar to the Tichonov approach replacing v_{t+1} by $\tilde{d}_t$. Can use linear solver.

5 Case-study

5.1 System general data

The numerical experiments described in this report were carried out considering instances of multi-stage linear stochastic problems based on an aggregate representation of the Brazilian Interconnected Power System long-term operation planning problem, as of January 2015. This system can be represented by a graph with four generation nodes – comprising sub-systems Southeast (SE), South (S), Northeast (NE) and North (N) – and one (Imperatriz, IM) transshipment node (see Figure 4).

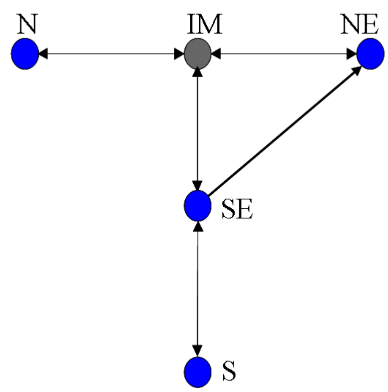

Figure 4: Aggregate representation of the Brazilian interconnected power system

The load of each area must be supplied by local hydro and thermal plants or by power flows among the interconnected areas. A slack thermal generator of high cost that increases with the amount of load curtailment accounts for load shortage at each area (Table 1). Interconnection limits between areas may differ depending on the flow direction, see Table 2. The energy balance equation for each sub-system has to be satisfied for each stage and scenario. There are bounds on stored and generated energy for each sub-system aggregate reservoir and on thermal generations.

	% of total load curtailment	Cost
P1	0 – 5	1420.34
P2	5 – 10	3064.15
P3	10 – 20	6403.81
P4	20 – 100	7276.40

Table 1: Deficit costs and depths ($/MWh)

The long-term planning horizon for the Brazilian case comprises 60 months, due to the existence of multi-year regulation capacity of some large reservoirs. In order to obtain a reasonable cost-to-go function that represents the continuity of the energy supply after these

		to				
		SE	S	NE	N	IM
	SE	–	7500	1000	0	4000
	S	5470	–	0	0	0
from	NE	600	0	–	0	3500
	N	0	0	0	–	∞
	IM	2940	0	3300	4407	–

Table 2: Interconnection limits between systems (MWave)

firsts 60 stages, a common practice is to add 60 more stages to the problem and consider a zero cost-to-go function at the end of the 120^{th} stage.

A scenario tree consisting of $1 \times N_2 \times N_3 \times \cdots \times N_{120}$ scenarios, for 120 stages, was constructed based on sampling of a periodic autoregressive multivariate statistical (PMAR) model with multiplicative error for the energy inflow record. In this (seasonal) model, the empirical distribution for each month and for every system is used to represent the corresponding noise distribution. The scenario tree is generated by sampling from these empirical distributions. The input data for this statistical model is based on 80 observations of the natural energy inflow (from year 1931 to 2010) for each of the considered 4 systems.

The study case general data, such as hydro and thermal plants data and interconnections capacities, were taken as static values throughout the planning horizon (120 months). The monthly seasonality of the demand is taken into account, that is, the energy inflows may vary along the stages but the annual value is constant. The total operation cost is the sum of thermal generating costs plus a penalty term that reflects energy shortage.

The objective function of the planning problem is to minimize the convex combination of the expectation and Average Value-at-Risk costs along the 120 months of the planning horizon, while supplying the subsystems load and obeying technical constraints.

5.2 Case study for objective function regularization

Alternative 1 consists in adding a quadratic regularization term to the objective function. This approach results in a unique solution for the stored volume and could stabilize the marginal cost. In order to evaluate the performance of this approach, a numerical experiment was performed.

For these experiments we have considered that:

- the regularization centers (see equation (43)) for each of the subsystems correspond to the monthly average values for the third year of the planning period obtained with a mean-AV@R policy;
- the observed past energy inflows before January 2010 and 2015 are representative for wet and dry hydrological years to condition the generation of the scenario tree;

- the objective function consists of the minimization of both the risk neutral and the mean-AV@R approaches, with and without the quadratic regularization term.
- constant values for quadratic penalty ϵ_t are equal to 0.0 (no regularization), 0.001, 0.01, 0.1, and 1.

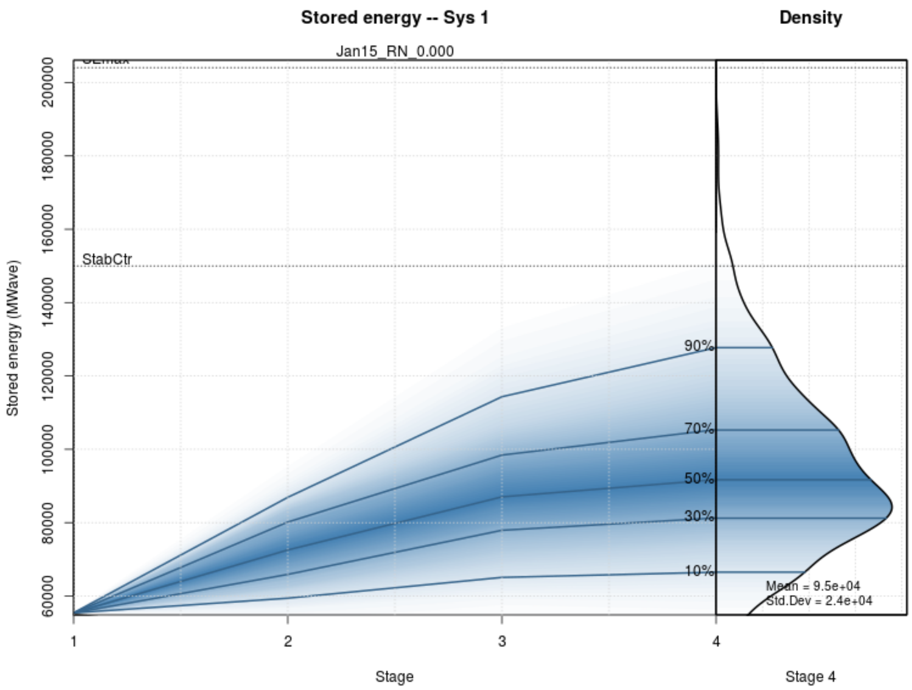

Figure 5

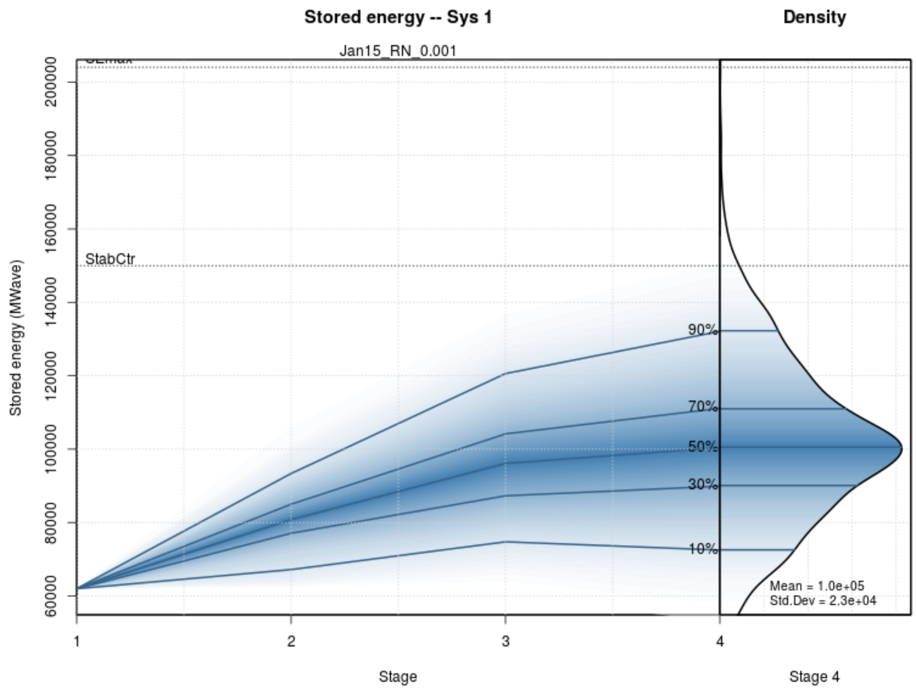

Figure 6

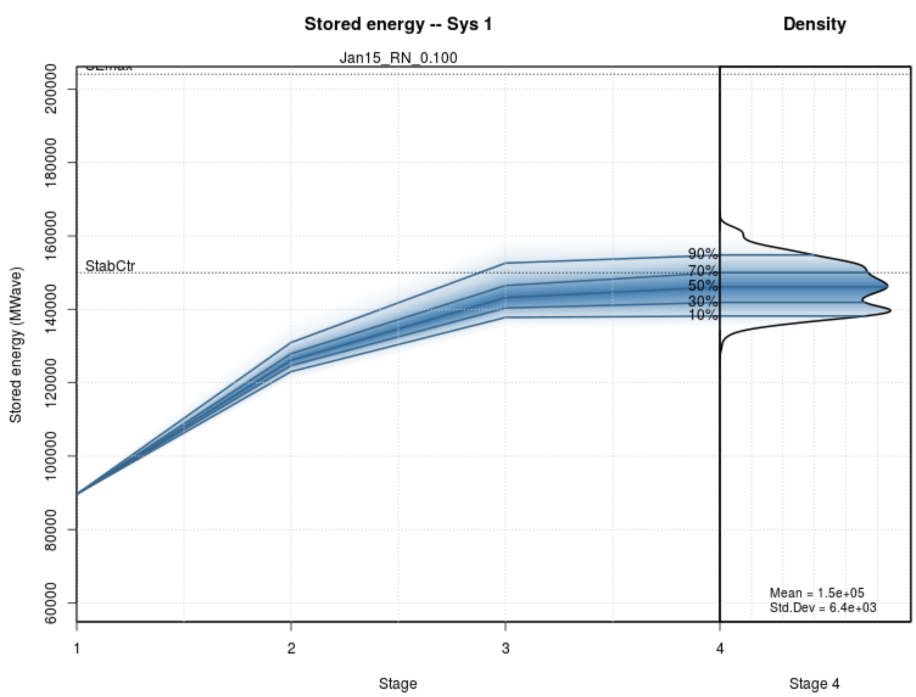

Figure 7

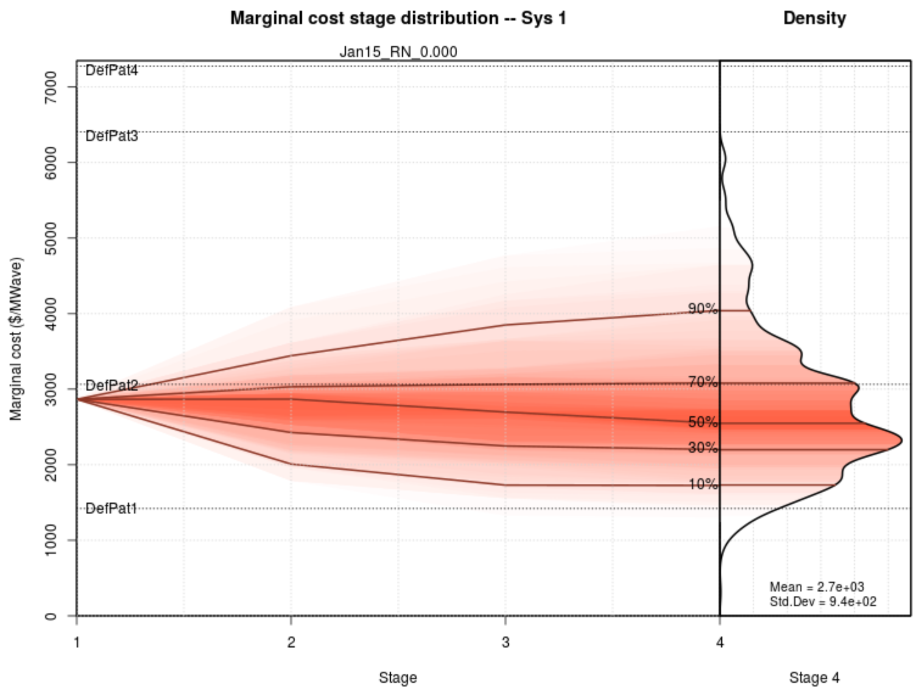
Marginal cost stage distribution -- Sys 1
Density
Jan15_RN_0.000
DefPat4
DefPat3
DefPat2
DefPat1
90%
70%
50%
30%
10%
Marginal cost ($/MWave)
0
1000
2000
3000
4000
5000
6000
7000
1
2
3
4
Stage
Stage 4
Mean = 2.7e+03
Std.Dev = 9.4e+02

Figure 8

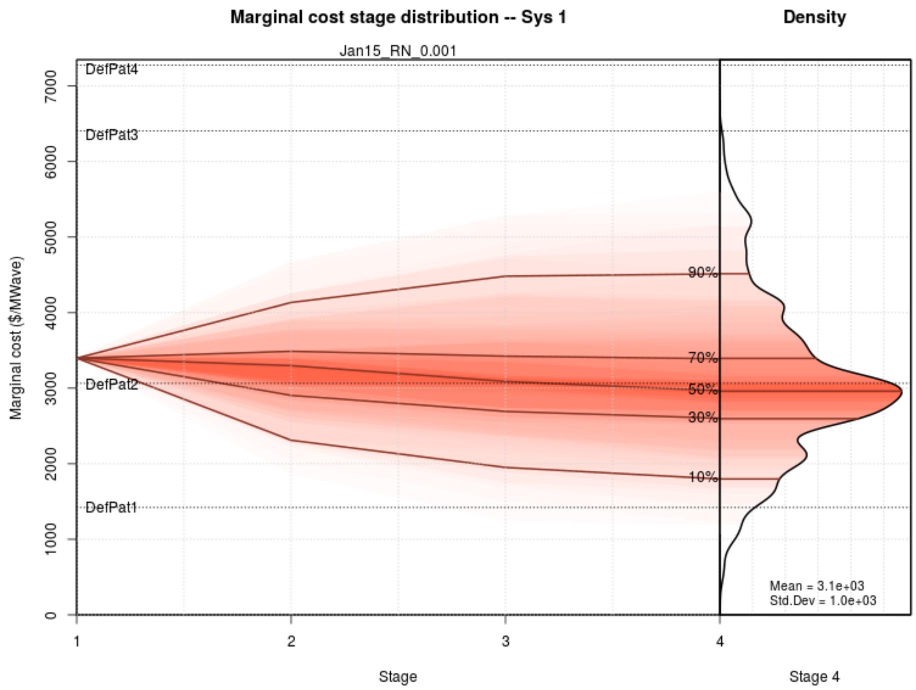

Figure 9

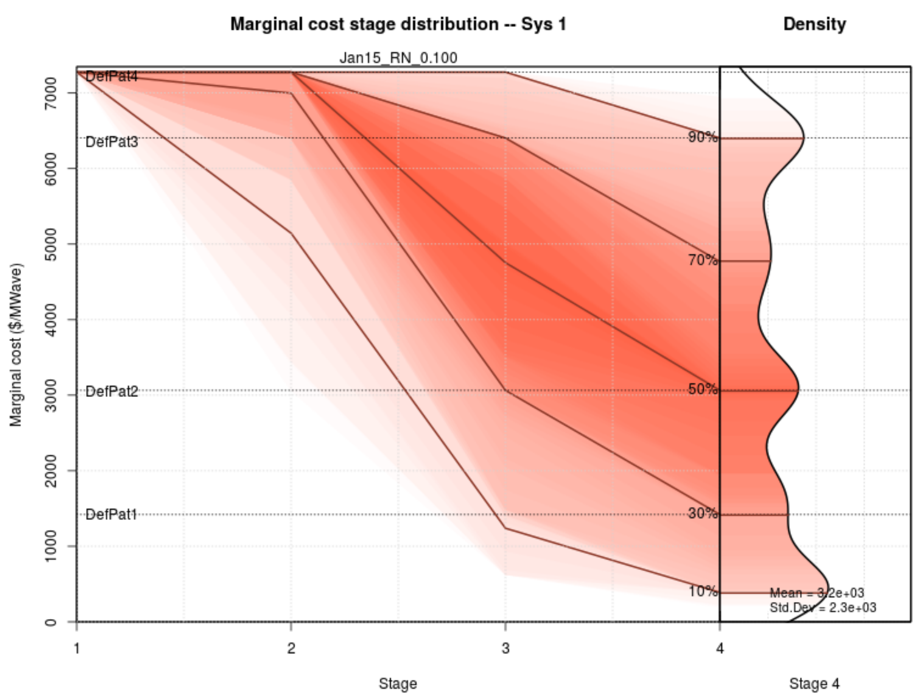

Figure 10

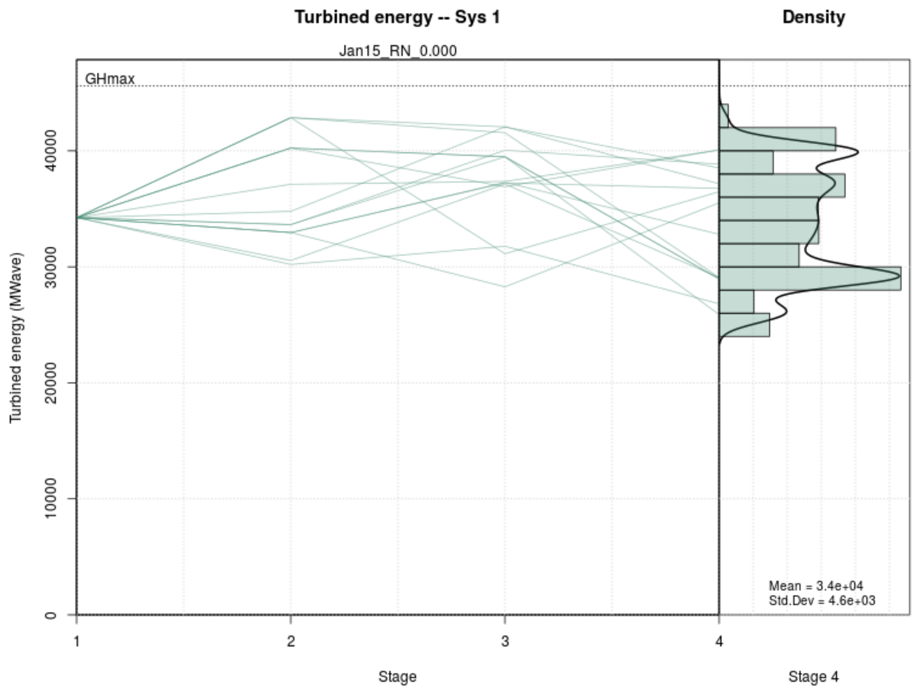

Figure 11

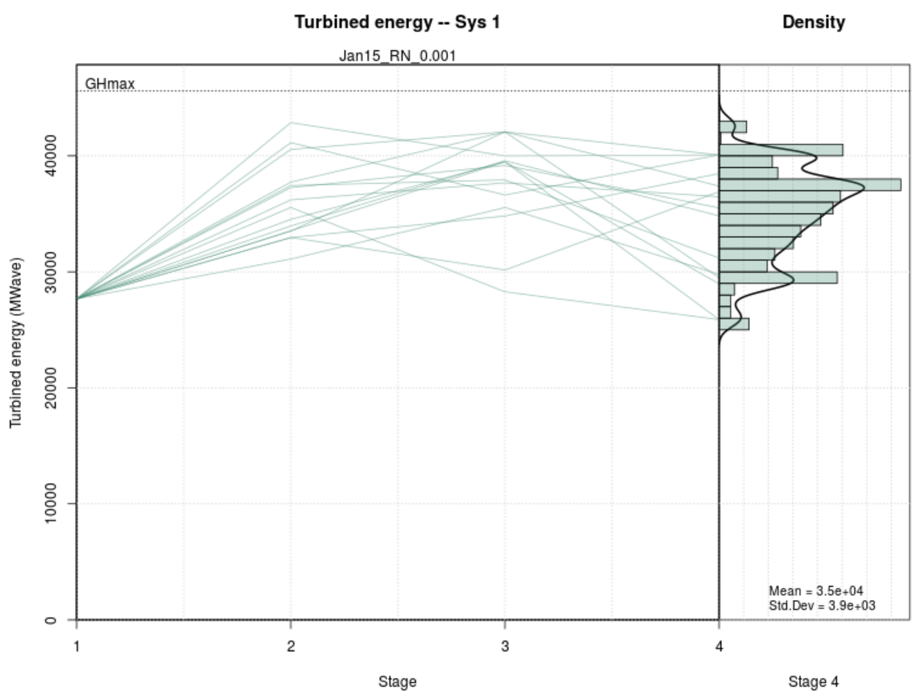

Figure 12

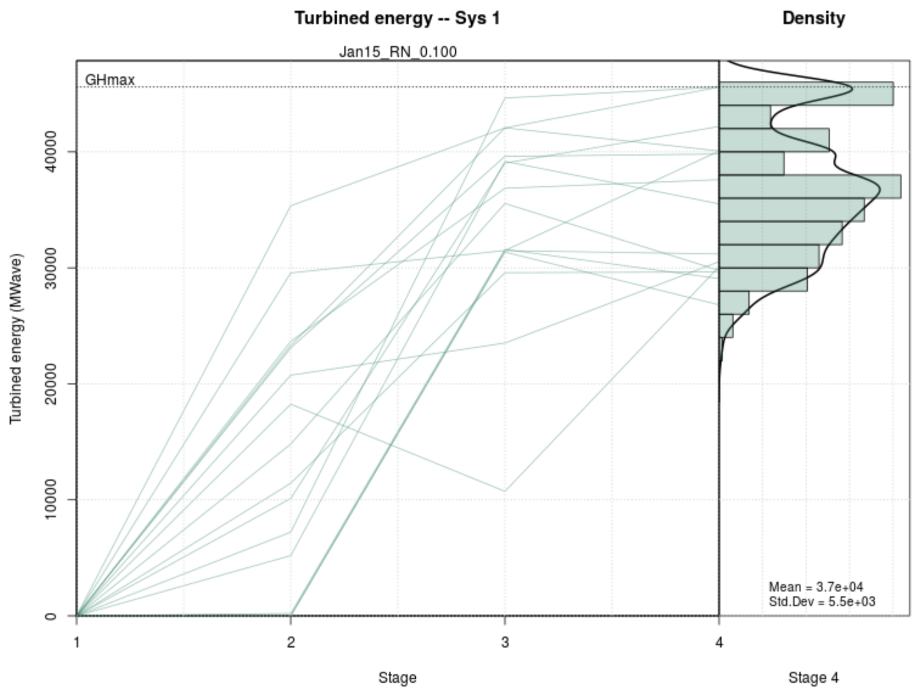

Figure 13

The main purpose of these experiments is to evaluate the additional effect of imposing the regularization on storage over the conditional marginal cost standard deviation (volatility);

Tables 3 and 4 summarizes the results for the mean marginal costs values of February considering 3000 out-of-sample scenarios for both risk neutral and risk averse measures.

Tables 5 and 7 summarizes the results for the volatility of February for both risk neutral and risk averse measures.

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Hydro Trend	Initial Month	February Marginal Cost mean quadratic penalty ϵ				
		0.0	0.001	0.01	0.1	1.00
dry	Jan/2015	2758.799	3223.820	5756.808	6688.187	7211.531
average	Jan/2009	1545.376	1644.911	1953.679	2005.091	2142.996
wet	Jan/2010	339.5925	283.2129	205.8631	288.3215	430.1468

Table 3: Mean marginal cost smoothing using quadratic regularization for Southeast Energy Storage — Risk Neutral case

Hydro Trend	Initial Month	February Marginal Cost Mean quadratic penalty ϵ				
		0.0	0.001	0.01	0.1	1.00
dry	Jan/2015	2758.799	3223.820	5713.563	6668.433	7219.365
average	Jan/2009	1546.246	1629.906	1953.679	2024.113	2208.203
wet	Jan/2010	336.7697	284.2375	215.4217	278.9300	433.7508

Table 4: Mean marginal cost smoothing using quadratic regularization for Southeast Energy Storage — Risk Averse case

Hydro Trend	Initial Month	February Marginal Cost Volatility quadratic penalty ϵ				
		0.0	0.001	0.01	0.1	1.00
dry	Jan/2015	593.4140	682.6321	1302.3617	1196.5978	509.6505
average	Jan/2009	282.3445	434.4033	925.5148	1584.2764	2206.3415
wet	Jan/2010	199.2158	159.7219	136.9821	264.3852	547.9454

Table 5: Volatility smoothing using quadratic regularization for Southeast Energy Storage — Risk Neutral case

	RHO_0	RHO_10-3e	RHO_10-2e	RHO_10-1e	RHO_10-0e
Jan15	593.41	682.63	1302.36	1196.60	509.65
Jan09	282.34	434.40	925.51	1584.28	2206.34
Jan10	199.22	159.72	136.98	264.39	547.95

Table 6: This table is just for conference purposes.

Hydro Trend	Initial Month	February Marginal Cost Volatility quadratic penalty ϵ				
		0.0	0.001	0.01	0.1	1.00
dry	Jan/2015	593.4140	682.6321	1312.6890	1214.1774	447.9946
average	Jan/2009	276.9192	427.5261	925.5148	1582.4752	2223.0841
wet	Jan/2010	195.7699	150.9643	154.3371	248.5328	494.8537

Table 7: Volatility smoothing using quadratic regularization Southeast Energy Storage — Risk Averse case

6 Conclusion

This report addressed the marginal cost variability in the solutions of the long-term planning operation model provided by the use of the SDDP algorithm. Previous studies concerning the uncertainty representation have already been done and were not dealt with here. Instead, we considered two alternatives to investigate the possible effects on marginal cost variability due to the convergence criteria and the discretization used to represent the scenario tree.

The first alternative proposed consists in adding a quadratic function to the total cost, so that the resulting convex function have only one optimal solution.

The second alternative considers a quadratic regularization of the immediate cost function with respect to the demand variation. This can be achieved by adding a regularization term to the immediate cost function, and in this document the corresponding formulation was fully described. Future computational studies can be performed to evaluate the performance of this alternative to smooth the marginal costs due to demand variation.

Additionally, it is also advisable to perform further studies regarding the uncertainty model formulation.

A An alternative approach to define Marginal Costs

We may suggest the following alternative way to definition of marginal costs, in both the risk neutral and risk averse cases. Let $\bar{x}_t = \bar{x}_t(b_{[t]})$ be a considered policy, say defined by computed approximations of the cost-to-go functions. In particular $\bar{x}_2 = \bar{x}_2(b_2)$ is obtained by solving problem (19) in the risk neutral case and problem (26) in the risk averse case, for given $x_1 = \bar{x}_1$ and an approximation of the cost-to-go function $\mathcal{Q}_3(\cdot)$. The solution $\bar{x}_2 = \bar{x}_2(b_2, d)$ of the respective second stage problems can also be considered as a function of the demand $d = L_2$. The cost $c_2^{\mathsf{T}} \bar{x}_2$ at the second stage is a function of b_2 and d. Let us denote this cost by $C_2(b_2, d)$, i.e.,

$$C_2(b_2, d) = c_2^{\mathsf{T}} [\bar{x}_2(b_2, d)]. \tag{47}$$

For small perturbations of the demand vector, say changing d to $d + h$ with small values of vector h, we can compute the corresponding perturbations of the second stage cost

$$C_2(b_2, d + h) - C_2(b_2, d). \tag{48}$$

If $\bar{x}_2(b_2, d)$ is differentiable in d, then partial derivatives $\frac{\partial C_2(b_2,d)}{\partial d_i}$ of the second stage cost can be evaluated by respective finite differences. For a given perturbation of the demands, by generating a sample of random vector b_2, the distribution of such marginal cost can be evaluated and the expected value can be estimated by the average.

A.1 Numerical experiments, risk averse case

Remark 1 *The purpose of this section is to illustrate the use of the above described two approaches to evaluate the marginal cost: the usual one, use of Lagrangian multipliers, and the alternative one, the simulation approach.*

A.1.1 The usual approach

The cost-to-go functions $\mathcal{Q}_{t+1}(x_t)$, of the form (24), were approximately evaluated by the SDDP algorithm and the first stage decision vector $x_1 = \bar{x}_1$ was computed by solving the corresponding first stage problem (25). Consequently $M = 3000$ samples b_2^j, $j = 1, ..., M$, of random vector b_2 were generated. For each generated vector b_2^j and given demand vector $\bar{d} = L_2$, the second stage problem of the form (19), with $x_1 = \bar{x}_1$ and the approximate risk averse cost-to-go function $\hat{\mathcal{Q}}_3(x_2)$, was solved. The computed optimal values of these problems give a sample of $M = 3000$ realizations of the random variable $\vartheta(b_2, \bar{d}) = Q_2(\bar{x}_1, b_2)$.

As a comparison between different seasons, we investigated two cases. We set January and September as the first stage respectively. The corresponding histograms are given below.

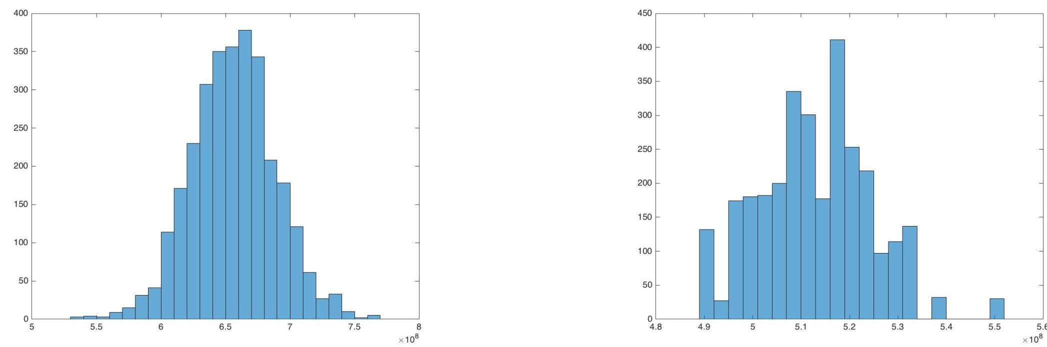
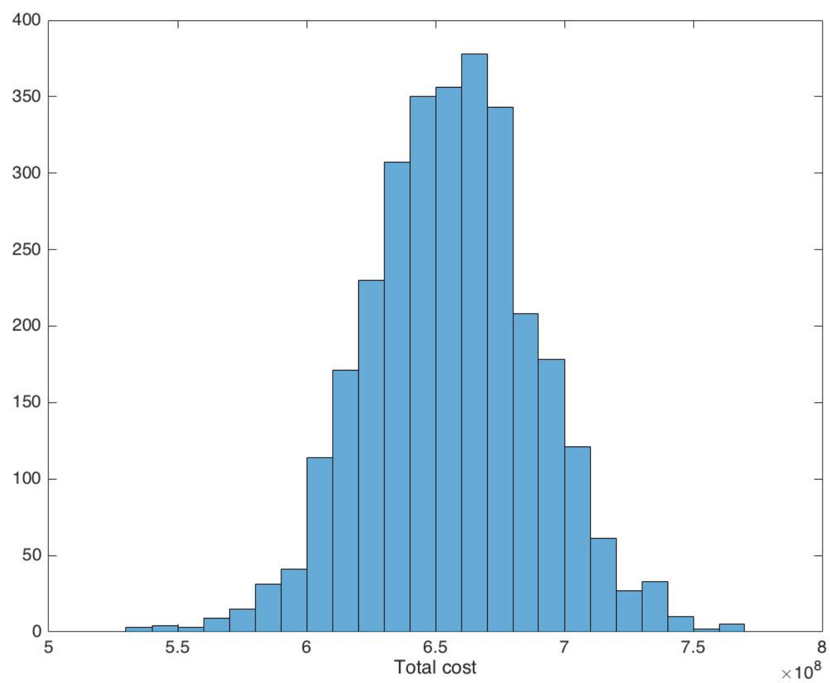


Figure 14: Histogram of the optimal values in two cases

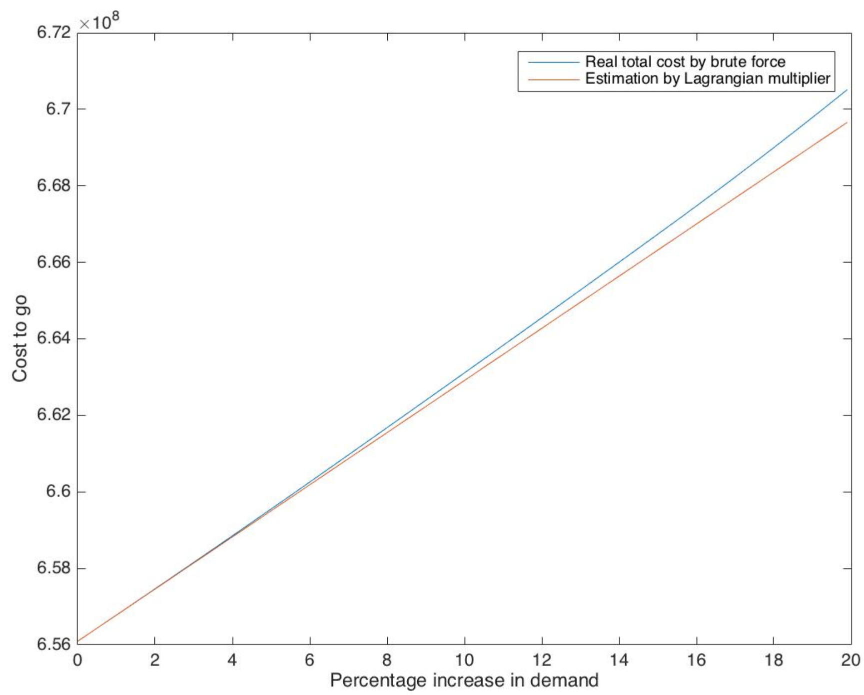


Recall that in the risk neutral case, for given demand d the random variable $\vartheta(b_2, d)$ represents the optimal expected total cost conditional on $x_1 = \bar{x}_1$ and b_2. In the considered risk averse case an interpretation of $\vartheta(b_2, d)$ is more involved and was discussed in the previous section.

Suppose now that the second stage demand is increased (proportionally for each component of the demand vector) by the following percentages: 0.1%, 0.2%,...,2%, i.e., the new demand is

$$d = 1.001 \times \bar{d},\ d = 1.002 \times \bar{d}, ..., d = 1.02 \times \bar{d}, \tag{49}$$

where $\bar{d}$ is the nominal (specified) value of the demand vector. Of course this results in the increase of the optimal value $\vartheta(b_2, d)$ of the corresponding problem (19). As it was discussed in the previous section, we can approximate this increase by using vector $\bar{\lambda}(b_2, \bar{d})$ of Lagrange multipliers (with minus sign) for the nominal value $\bar{d}$ of the demand vector, provided this vector of Lagrange multipliers is unique. In the graph below are plotted the average (over the 3000 sample replications of b_2) of the increase in $\vartheta(b_2, d)$, that is $M^{-1}\sum_{j=1}^{M}\vartheta(b_2^j, d)$, and the corresponding linear approximation based on the Lagrange multipliers vector. It can be seen that these relatively small changes in the demand the linear approximation was very good. This experiment was performed for the month of January at the first stage.



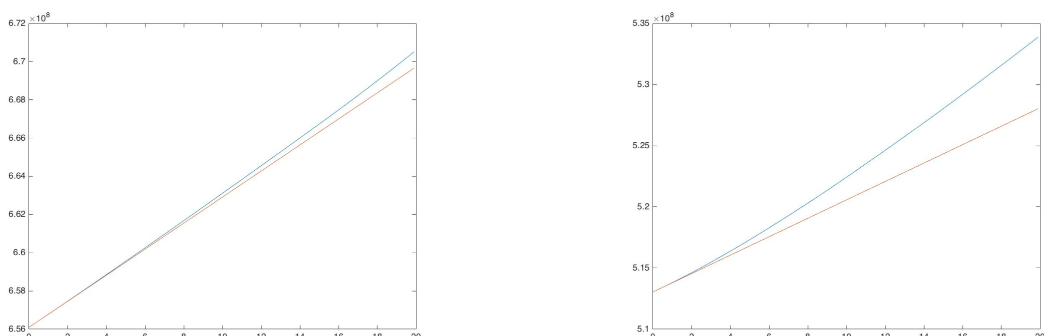


Figure 15: Sensitivity to demand of the cost-to-go (in blue) and its linear approximation by Lagrange multipliers (in red) in two cases

A.1.2 The alternative approach

Consider the alternative approach discussed in section A. For each generated b_2^j, $j = 1, ..., 3000$, and the policy defined by the application of the SDDP algorithm, the cost $C_2(b_2^j, d)$, defined in (47), was computed. The histograms of these costs for the nominal value of the demand vector are shown below.

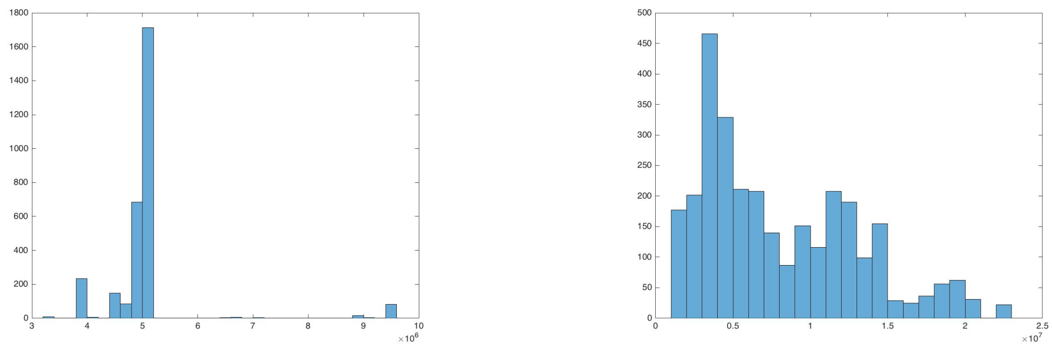


Figure 16: Histograms of the second stage costs in two cases

Consider the average changes $\mathbb{E}[C_2(b_2, d) - C_2(b_2, \bar{d})]$ of the second stage costs for small changes of the demand vector, with demand changes specified in (49). This can be estimated by averaging the corresponding values of the second stage costs. These average changes of the second stage costs as a function of the respective changes of the demand vector are shown in the following plot.

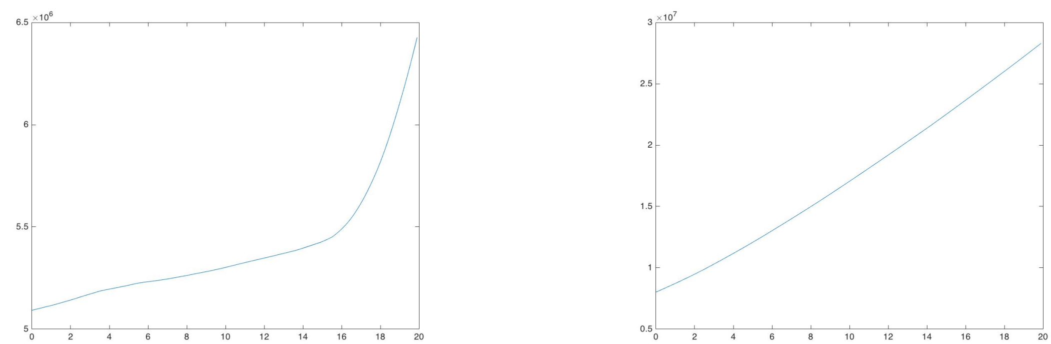


Figure 17: Sensitivity to demand of the second stage costs in two cases

References

ONS. Avaliação da qualidade da solução da PDDE com o uso de CVAR e independência temporal no cálculo da função de custo futuro. Technical Report NT-140/2013, ONS, November 2013.

Mario Veiga Ferraz Pereira. Comments on SDDP algorithm numerical performance. Personal communication, 2008.

R. Tyrrell Rockafellar and Roger J.-B. Wets. *Variational Analysis*, volume 317 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 1 edition, 10 2011. ISBN 9783540627722. doi: 10.1007/978-3-642-02431-3.

M. P. Soares, A. Street, and D. M. Valladão. On the solution variability reduction of stochastic dual dynamic programming applied to energy planning. In *2014 IEEE PES General Meeting — Conference Exposition*, pages 1–5, July 2014. doi: 10.1109/PESGM.2014.6939356.

Steven Stoft. *Power System Economics: Designing Markets for Electricity*. Wiley-IEEE Press, 1 edition, 5 2002. ISBN 9780471150404.