OPTIMAL PLANNING FOR ELECTRIFICATION OF PUBLIC TRANSIT SYSTEMS AND INFINITE-DIMENSIONAL LINEAR PROGRAMMING

A Dissertation
Presented to
The Academic Faculty

By

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
H. Milton Stewart School of Industrial and Systems Engineering
College of Engineering

Georgia Institute of Technology

April 2023

OPTIMAL PLANNING FOR ELECTRIFICATION OF PUBLIC TRANSIT SYSTEMS AND INFINITE-DIMENSIONAL LINEAR PROGRAMMING

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Hope lies in dreams, in imagination, and in the courage of those who dare to make dreams into reality.

Jonas Salk

To those who pursue their dreams despite the odds.

ACKNOWLEDGMENTS

I am fortunate to have had the opportunity to work with my advisor, professor Andy Sun, whose mentoring and collaboration have been instrumental in my personal and professional growth. In addition to teaching me research skills and paper writing, he has helped me become more professional and strive for excellence, recognize the significance of a big-picture perspective, and develop many other skills that have become integral parts of my personality. Over time, we have established a collaborative relationship built on mutual feedback, improved communication, and expectations, which I sincerely appreciate.

I would like to thank my committee members, professor Santanu Dey, professor Daniel Molzahn, professor Edwin Romeijn, and professor Alexander Shapiro for making the time to participate in my committee. Their suggestions and feedback had a significant impact on this work. I also would like to express my gratitude to professor Alan Erera for admitting me to the Operations Research Ph.D. program at Georgia Tech.

My deepest appreciation goes to my wife and best friend, Liliane Portugal. I recognize and appreciate your bravery and willingness when you temporarily set aside your professional career to support me in pursuing mine. You remained a constant source of support, offering encouragement and emotional assistance when I needed it most. You have brought immense joy and excitement to my life, and I am incredibly honored to be your husband. You can always count on my unwavering support and affection.

I am grateful to my mother and sister for their steadfast love and care, which have been the bedrock of my achievements. Their investment in my education until now has been invaluable, and I appreciate it immensely. I have always admired their perseverance and determination to succeed in life.

I am fortunate to have Dr. Antonio Figueiredo, my therapist, to support me mentally during the most tumultuous moments of my doctoral studies and the pandemic. I feel privileged to have received the assistance of such an exceptional professional, which has contributed to my personal growth and happiness. Thank you for being present during the most challenging phases of my journey.

A special thanks go to my friend Arnesh Sujanani for cheering me up and providing personal help since the beginning of my Ph.D. I would like to acknowledge his loyalty as a friend and for sharing both the most joyful and challenging moments throughout my Ph.D. journey.

I want to extend my heartfelt gratitude to my long-time friends and mentors: Claudio, Joari, Bernardo, and Evandro. Their advice throughout the years have been invaluable, and I cherish their support deeply.

I am greatly thankful to professor Alexander Shapiro and the late professor Shabbir Ahmed for their exceptional textbooks and journal papers that motivated me to pursue the field of Operations Research. Their trust in my abilities and recommendation letters were pivotal in my admission to the Operations Research program at ISyE-Gatech. I will always remain grateful to them for their guidance and inspiration.

A special thanks go to Amanda Ford and Dr. Dawn Strickland for their invaluable assistance in keeping me on track with the bureaucratic requirements of the Ph.D.

I would like to express my gratitude to my friends Aster and Denise for welcoming me at the start of my doctoral studies and for providing continuous support to remain focused and optimistic. My sincere appreciation also goes out to my colleagues Sushil, Prachi, Kibria, Ritesh, Nidhima, Selin, Jade, Dipayan, and Julie, along with everyone who made my Ph.D. journey more manageable and enjoyable.

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SUMMARY

Worldwide, the transportation sector is the second largest contributor to greenhouse gas emissions (GHG) after the electricity industry, while in the US, it is the largest contributor. One attractive solution to curb the GHG emission in the public transit segment is the Battery Electric Bus (BEB). A BEB produces zero tailpipe GHG emissions, its fuel cost is around 40% cheaper than a similar-sized fossil-fuel bus, its noise level is significantly lower, and it has less maintenance need. However, BEB technology poses new challenges to the bus operation planning and charging placement due to the limited travel range and long charging hours. The optimal charging infrastructure plan may also vary with the spatial distribution of the routes.

First, we propose an Optimal Planning of Charging Facilities and Electric Bus Fleet (OPCF-EBF) model to optimally plan the transition to an entirely electric bus fleet. The OPCF-EBF model brings together long-term planning investment decisions and short-term operation assessment, where the latter considers modular arithmetic to model the repeating of daily 24-hour bus demand in each planning year. The proposed OPCF-EBF model is shown to be NP-hard through reduction from the uncapacitated facility location problem. We propose an effective and computationally scalable primal heuristic called "Policy-Restriction" that significantly outperforms and improves Gurobi. We conduct extensive computational studies using real-world data from public transit systems in major cities in the U.S. and around the world, which reveal insights into the optimal investment and operational strategies.

Second, we explore another dimension of the OPCF-EBF model with only one bus route and only depot BEBs, but an arbitrary number of battery states. We call this a fleet-sizing problem. We show that, under a simple non-preemptive charging strategy with no early charging and idling, the fleet-sizing problem is polynomially solvable. The proof of this fact relies on the "almost" total unimodular nature of the constraint matrix for the operation problem given the number of depot BEBs. We also rely on a proximity result that quantifies

the distance of the optimal solution of a Separable Convex Integer Program and the solution of its corresponding linear relaxation. We then prove the polynomial-time complexity based on the number of intermediate auxiliary linear programs necessary to obtain an integral optimal solution. Inspired by the fleet-sizing problem, we introduce a special class of polynomially solvable Separable Integer Programs followed by the notion of a Dyad Contiguous Row (DCR) matrix.

Third, we investigate more deeply the notion of a stationary optimization problem. We introduce a type of primal and dual for infinite-dimensional stationary linear programs based on a restriction to ℓ_∞ and ℓ_1 spaces of appropriate dimensions. We motivate this approach from the fixed-point formulation of discounted stationary programs and prove weak duality. We illustrate with a hydro-thermal stationary planning problem that strong duality may hold for a large class of infinite-dimensional stationary linear programs. In particular, the value function of the latter is piecewise linear convex with countably many affine functions. Weak duality may fail if the ℓ_∞ and ℓ_1 set constraints are removed.

Lastly, we investigate an algebraic method to characterize extreme points of a class of infinite dimensional optimization problems called row-finite linear programs. We introduce the notion of an Asymptotically Compatible (AC) solution to extend the definition of a basic solution for infinite dimensional linear programs. In fact, a solution to a row-finite inequality system is extreme if the only AC solution to the system induced by the active linear constraints is the trivial one. We describe how the Gauss-Jordan elimination algorithm parameterize all the solutions of row-finite equality system. Our approach is illustrated in the characterization of extreme circulations over infinite digraphs of finite degree.

CHAPTER 1
INTRODUCTION

1.1 Background

Scientists are researching decarbonization solutions in every segment of society for the next challenge regarding human existence: climate change. According to [1], more than 75% of the Greenhouse gas (GHG) emissions in the US come from power production, industrial activities, and the transportation sector, where the latter is currently the largest GHC emitter. There are a couple of initiatives in the transportation segment regarding zero tailpipe emissions. Still, as of now, battery electric vehicles (EVs) offer the best combination of location convenience to refuel a car and direct consumption of clean energy from renewable sources [2].

However, EVs pose additional challenges in comparison with fossil fuel vehicles. For example, the travel range of an EV is considerably lower due to limitations on the current battery storage [3, 4]. Also, the charging time can be significantly longer, which might be inconvenient for long-range trips. The success of a transition to entire electric fleets requires extensive planning, and the best solution depends on the specific application. This thesis focuses on supporting decision tools for public transportation, explicitly to aid the transition from conventional bus fleets to battery electric fleets on public transit systems. Other EV applications, such as private vehicles or freight logistics, are outside the scope of this work. For a review on EV applications, see [4].

We propose a mixed-integer program for the optimal transition to an entirely electric bus fleet regarding the practical needs of a public transit agency. Buses have life cycles of 12 years on average, and transit agencies can only retire some of their buses at a time. Our model considers a yearly planning horizon for charging infrastructure and fleet renewal investment with bus retirement targets, charging location, and budget constraints.

We also propose a realistic operation model to assess the fleet operation and costs each year. The idea is to use the bus schedules informed through the General Transit Feed Specification (GTFS) file regarding each public transit system to extract information about the routes and associated bus demand for each hourly time interval. Our operation meets the bus demand using a mix of electric and conventional bus fleets along the transition.

The operation model also offers insights into our model's operation peaks and charging dynamics. It represents a stationary bus schedule for a regular weekday, e.g., Monday. We assume a 24-hour operation is periodic and repeats throughout the year with the coupling constraints between the variables of the last and first 24 hours. To this end, our time indexes are equivalence classes under congruence modulo 24, which has the wrap-around effect when we perform addition and subtraction operations.

We consider two types of Battery Electric Buses (BEB) that significantly differ in function. The depot BEBs rely on their large battery capacity to operate and return to the bus depot to recharge for a few hours. On the other hand, the on-route BEBs can operate with a smaller battery storage and charge whenever they reach their bus terminal. So, on-route BEBs can work similarly to conventional buses since they can accommodate their charge into the bus driver layover time. However, the charger infrastructure and energy costs are much higher for on-route BEBs than depot BEBs. Therefore, it is often unclear if one approach is better without proper simulation and optimization.

The computational complexity of our Optimal Planning of Charging Facilities and Electric Bus Fleets (OPCF-EBF) model is NP-Hard, and its proof is a polynomial-time reduction from the Uncapacitated Facility Location (UFL) problem. In practice, the OPCF-EBF is also a numerically challenging problem. Even Gurobi cannot not find a solution with an average gap smaller than 52% within 4 hours of computation in a cluster with 86 cores. We proposed a scalable primal heuristic that accelerates the search for an excellent primal solution, outperforming Gurobi in most real cases.

Another dimension of our analysis is in terms of the operation model. We investigate whether the scheduling of the bus charging, operation, and fleet sizing could be another cause for numerical issues. The high dimensionality of our model is primarily explained by the different configurations one could schedule the fleet charging.

We investigate the simplest fleet sizing and operation model, which assumes only one route, unlimited charging capacity, and depot BEBs. In terms of operation, the depot BEBs can only charge when depleted. Once fully recharged, they have to resume operation immediately. This model is polynomially solvable, but the proof of such a fact is nontrivial. Indeed, we reformulate our problem as a two-stage model, in which the first stage contains only the fleet-sizing variables and constraints, while the second stage is the operation given the fleet size. We prove that the second stage problem is integral, i.e., the linear relaxation is the convex hull of the integral feasible set, despite the second stage problem's constraint matrix not being totally unimodular.

We frame our two-stage integer program as a Separable Convex Integral Program (SCIP). The novelty of our polynomial-time reduction lies in using a proximity theorem [5] for SCIPs to limit the search for an optimal integral solution. This analysis only works for instances where we do not allow either idle buses or early bus charging. For more flexible operations, the computational complexity remains open. Then, we generalize our model to a class of two-stage Separable Integral Programs. We introduce the Dyadic Contiguous Row (DCR) matrix that generalizes the notion of a row circular matrix [6] and contains our second-stage operation model's constraint matrix as a particular case. Our polynomial-time algorithm is based on the proximity theorem for SCIPs can solve this class of mixed-integer programs as well.

Motivated by the stationary operation of our bus schedules, we also investigate the meaning of a stationary linear program more deeply. More precisely, we depart from a fixed point value function perspective and introduce the elements of a stationary infinite-dimensional linear program. The first challenge is to guarantee convergence of the discounted series that naturally arises in the objective function. One could take several approaches to make the objective well-defined, such as taking the series's liminf [7] or assuming a uniform bound for the decision variables [8, 9]. Indeed, we choose a balance between those two. We introduce the ℓ_∞ set constraint in defining a stationary infinite-dimensional linear program. Our approach preserves the objective's linearity property and is less restrictive than the uniform bounds on the variable space.

Following this analogy, it is surprisingly simple to define a dual program. We apply the same duality rules as in a finite-dimensional linear program and add the ℓ_1 set constraint. Weak Duality follows from simple algebraic manipulations and Fubini's theorem for absolutely convergent series. We provide a toy problem inspired by a hydro-thermal power generation planning problem that supports our primal-dual setting and satisfies Strong Duality. We also observe with an example that by dropping the ℓ_∞ and ℓ_1 set constraints, Weak Duality may fail. However, once we enforced those set constraints, the same example satisfies Strong Duality. Those pieces of evidence support the claim that Strong Duality may hold for a large class of stationary infinite-dimensional linear programs.

Since we have explicit primal and dual optimal solutions for the stationary hydro-thermal planning problem, the natural follow-up question is whether or not those solutions are extreme points. Unfortunately, there is a gap in the literature regarding basic feasible solutions for finite and infinite-dimensional linear programs [10, 11]. This motivated our investigation for an algebraic characterization of extreme points that could serve for computations similar to the finite-dimensional counterpart. Given a feasible solution, we introduce the notion of an asymptotically compatible (AC) vector, which connects with the idea of a feasible direction. Indeed, we prove that a direction is AC with a feasible solution if, and only if, the direction and the opposite sign direction are both feasible at the same point.

The asymptotically compatible concept is central to extending the notion of a basic feasible solution to any convex set defined by arbitrary linear constraints. Indeed, a feasible solution is extreme if, and only if, the linear equality system induced by the binding constraints has a unique AC solution, which is the trivial one. We observe from our definition that the set of AC vectors form a vector subspace. Using our basic feasible solution characterization, we prove that the primal and dual optimal solutions to the hydro-thermal planning problems are extreme points. We then describe a general method, called the Gauss-Jordan method, to find all the solutions for binding linear equality systems called the row-finite linear systems. Finally, we illustrate this technique in an extreme point example.

As a more general application, we provide alternative proof of the extreme point characterization for network flows on infinite graphs of finite degrees [11]. Recall that a flow in a finite digraph is extreme if, and only if, the residual graph has no weak cycle. Such a condition is only necessary for a digraph with a countable set of nodes and a finite degree in each node. Indeed, we prove that a flow is extreme if, and only if, the residual graph has no weak cycle and there are not two arc-disjoint trees with positive max-residual capacity at a common node. Intuitively, we cannot reroute flows from "infinity" in one tree into another. Our extension of a basic feasible solution is a simplifying tool for the extreme flow characterization of [11].

1.2 Contributions

In Chapter 2, we develop a realistic model for the fleet sizing and charging infrastructure planning for Battery Electric Buses (BEBs). Below, we summarize our contributions based on each category:

1. **Modeling**: Our Optimal Planning of Charging Facilities and Electric Bus Fleet (OPCF-EBF) model for public transit systems has two distinct features:
    (a) The two-time-scale structure of the OPCF-EBF model brings together long-term planning and short-term operation, with annual investment decisions of charging infrastructure and fleet composition over a decade-long transition horizon and hourly operation decisions of bus charging and scheduling over 24

hours in each planning year.

(b) Modular arithmetic is used to model the repeating of daily 24-hour bus demand in each planning year. Various charging strategies, such as early charging (charging before battery full depletion), idling (neither working nor charging), non-preemptive charging (charging until full), are modeled, together with practical planning strategies such as utilization of existing bus depots and route terminals as potential charging facilities, respecting of retirement schedules of conventional buses, and various realistic investment and operational costs.

2. **Characterizations**: We characterize the computational complexity of the proposed model. The proposed OPCF-EBF model is shown to be NP-hard through reduction from the uncapacitated facility location problem. Even with one planning period and two charging states for depot-charged BEBs, the OPCF-EBF model is still NP-hard, as the numbers of bus routes and charging depots grow.

3. **Algorithm**: We propose an effective and computationally scalable primal heuristic called "Policy-Restriction" that significantly outperforms and improves Gurobi.

4. **Real-world case studies**: We conduct extensive computational studies using real-world data from public transit systems in major cities in the U.S. and around the world, which reveal insights into the optimal investment and operational strategies. For example, an optimal investment decision tends to invest in depot chargers before on-route chargers; an optimal operational solution tends to use on-route BEBs to meet base-load bus demand and to use depot BEBs to meet peaking bus demand. These empirical insights are also explained through theoretical analysis.

In Chapter 3, we explore another dimension of the OPCF-EBF model with only one bus route and only depot BEBs, but an arbitrary number of battery states. We call this a fleet-sizing problem. The goal is to prove that, under a simple non-preemptive charging strategy with no early charging and idling, the fleet-sizing problem is polynomially solvable. For this, we develop the following techniques:

1. **Tight LP relaxation of the operation problem:** The coefficient matrix of the operation problem may not be totally unimodular (TU). However, interestingly, by exploiting the rich symmetry imposed by the non-preemptive charging policy and the modular arithmetic, we can reformulate and unimodularly transform the operation problem to an equivalent formulation that does have the TU property.

2. **The fleet sizing problem as a Separable Convex Integer Program:**

   (a) Using the previous result, we extend the value function of the operation problem to an extended-real-valued convex piecewise linear function. Thus, the fleet sizing problem is essentially a separable convex integer program (SCIP), separable over the investment periods.

   (b) Underlying this result is a proximity theorem proved for general SCIP that an optimal integer solution of SCIP belongs to the integer lattice of a ball centered at the LP relaxation's optimal solution. The key result shows that the search over the integer lattice can be further reformulated as a new integer linear program, which has an exact LP relaxation.

   (c) Finally in the last step, we bound the number and size of all the LPs involved, and refer to the arithmetic complexity of an algorithm of Vaidya [12] to conclude the polynomial-time complexity of the fleet sizing problem.

3. **The Dyadic Contiguous Row (DCR) matrix:** We introduce a special class of separable integer program with totally unimodular constraints that can serve as a two stage decomposition method to prove polynomial solvability. In fact, we define the notion of a Dyadic Contiguous Row (DCR) matrix which extends the definition of a row-circular matrix of Bartholdi [6]. Given that the second stage integer program

has a DCR coefficient matrix, the same complexity analysis performed for the fleet sizing problem applies.

In Chapter 4, we investigate a notion of duality for infinite dimensional *stationary* linear programs:

1. **A duality framework:** Based on the duality rules for finite dimensional LPs and restrictions to appropriate ℓ_∞ and ℓ_1 spaces, we introduce primal and dual infinite dimensional stationary linear programs and prove weak duality.

2. **Evidence of a strong duality result:** Strong duality may hold for a large class of problems in our infinite dimensional setting as illustrated by a stationary hydro-thermal power generation planning problem. Using a counter-example, we show that weak duality may fail if one disregards the ℓ_∞ and ℓ_1 set constraints. However, that same example satisfies strong duality when the ℓ_∞ and ℓ_1 constraints are enforced.

In Chapter 5, we investigate an algebraic method to characterize extreme points for convex sets defined by row-finite linear systems. This is motivated by the primal and dual optimal solutions of the stationary hydro-thermal power generation planning problem. The following are the main aspects of Chapter 5:

1. **Asymptotically Compatible vectors and basic feasible solutions:** The geometric definition of an extreme point may not be a convenient method to certify whether or not a given point of a convex set is extreme. Given arbitrary linear constraints, we extend the definition of a basic feasible solution using the notion of Asymptotically Compatible (AC) vectors. We show that a point is extreme if and only if the only *AC-solution* to the linear equality system induced by the set of active constraints is the trivial solution. This method provides a direct proof that the primal and dual optimal solutions of the stationary hydro-thermal power generation planning problem are extreme points.

2. **Row-finite linear systems:** A row-finite linear system over the sequence space of real numbers is a countable set of linear constraints induced by coefficients with a finite number of non-zero elements. We observe that the Gauss-Jordan elimination method of [13] can parameterize all the solutions of a row-finite *equality* system. Such method may certify the existence of a unique (or multiple) AC-solution for the set of active constraints', or, equivalently, that the corresponding feasible solution is (not) extreme. This idea is a direct parallel with the Gaussian elimination method for equality linear systems of finite dimension but it has the rightmost pivoting as an important distinction.

3. **Application to extreme flows over countably infinite graphs:** We illustrate the use of our extreme point result for an alternative proof of the extreme flow characterization in countably infinite graphs of finite node degree. The original result is from [11] and it provides another condition on the residual graph besides not having a cycle that form the necessary and sufficient conditions for a network flow to be extreme.

CHAPTER 2
AN OPTIMAL PLANNING MODEL FOR CHARGING FACILITY AND BATTERY ELECTRIC BUS FLEET

2.1 Introduction

Modern transportation relies heavily on fossil fuels. However, fossil fuel consumption endangers our world. According to the Intergovernmental Panel on Climate Change, [14], carbon dioxide emission is the predominant cause of global warming and has already increased the global average temperature by one degree Celsius above the pre-industrial revolution level. An increase above two degrees Celsius can cause extreme weather events, a shortage of food supply, and higher sea levels. The United States, among many other countries, aims to reduce its net greenhouse gases (GHG) emissions by 50% below the 2005 levels in the coming decade, according to [15].

To curb the GHG emission, the world is seeking alternative energy sources. Worldwide, the transportation sector is the second largest contributor to greenhouse gas emissions [16] after the electricity industry, while in the US, it is the largest contributor [1]. Although buses represent a fraction of the transportation segment, their effect on public health is significant, due to the fact that buses operate in densely populated urban areas and emissions such as nitrogen oxide and particulate matter adversely affects cardiovascular and respiratory health, see [17] and [18].

One attractive solution in the public transit segment is the Battery Electric Bus (BEB). A BEB produces zero tailpipe GHG emissions, its fuel cost is around 40% cheaper than a similar-sized fossil-fuel bus, its noise level is significantly lower, and it has less maintenance need. There has been an ever-growing number of transportation agencies all over the world switching to BEBs as a more sustainable option for public transportation [19].

However, BEB technology poses new challenges to the bus operation planning, fleet sizing, and the charging placement due to the limited travel range and long charging hours. The optimal charging infrastructure plan may vary with the spatial distribution of the routes. The minimum fleet size to maintain the same service level may increase to compensate for the charging time. Indeed, when planning the transition to an entirely electric bus fleet, one should consider the fleet sizing, charging placement, and the impact on bus operation altogether.

There are two typical charging technologies that are adopted on a commercial scale, namely depot and on-route charging. Depot charging refers to charging BEBs with a low-voltage alternating current (AC) system in a bus depot or garage, which has lower deployment and usage costs but requires several hours to fully charge a BEB. On-route charging, in contrast, uses a direct current (DC) fast charging system. It has a higher cost than a depot charger, but can be used in bus terminals for fast charging during the layover time of a bus to provide the energy needed for a round trip on a bus route.

In this thesis, we propose a novel integer linear optimization model for the joint optimal planning and operation of depot and on-route charging facilities and BEB fleets. This model plans the transition to an entirely BEB fleet through annual investment targets which consider the agency's budget, the conventional bus retirement targets, and the operation of the mixed fleet of BEBs and conventional buses in transition. The model is realistic in capturing periodic bus schedules, BEB charging dynamics, various investment and operational costs, with depot locations, bus routes, and bus demand extracted from real transit agency data.

2.2 Related work

The scientific literature on electric vehicles (EVs) has investigated a wide variety of modeling techniques and applications. In this section, we mention some recent papers that are related to electrical bus fleet planning and operations.

[20] presents a model for EV fleet renewal from the French national postal service. [21] use distributionally robust optimization to plan the battery swapping infrastructure. The work of [22] evaluates the implementation of battery swapping for EVs in freight logistics, while [23] consider the environmental impact of the adoption of electric passenger vehicles. [24] consider an EV routing model with a nonlinear charging function. [4] present a literature review of other models related to EV infrastructure planning, EV charging operations, and public policy in the EV industry.

The work of [25] considers battery charging scheduling, battery swapping, and the bus scheduling of a mixed bus fleet including battery-electric, compressed natural gas, diesel, and hybrid-diesel buses. [26] concentrate on fast charging scheduling of battery electric buses to minimize charging costs and power grid impact. [27] focus on battery capacity fade and propose an optimization model, which considers the reduction in the storage capacity of batteries, to schedule the electric bus charge. The works of [3], [28], and [29] consider a planning model for the composition of an electrical bus fleet using depot charge BEBs. With regards to technological factors, the paper of [30] assesses the impact of wireless chargers on battery-electric bus scheduling. [31] consider a hybrid solution of hydrogen and electric buses with the concept of multi-product charging stations.

With regards to modeling specific features, the papers by [32] and [3] propose models for installing fast chargers focused on the electric demand charge, which is the cost associated with the variations in power demand. [33] assess the potential reduction of the peak demand charge by installing energy storage units for on-route fast chargers. The works of [34] and [35] propose a fast charging location model for battery-electric buses, considering the bus operation and the power distribution. [36] propose a stochastic model for managing the electric bus charge, the photovoltaic energy production, and energy storage systems. [37] and [38] present a long-term multi-period model for the electric bus integration into urban bus networks.

It is also worth commenting on the diversity of modeling techniques associated with battery-electric buses. [39] use bi-level programming to formulate an electrical transit route planning problem, where the upper level determines the route structure and charging station location while the lower level calculates the user cost. [40] propose a charger location model that describes the bus charging using queuing theory. [41]'s work is based on a stochastic model for the interaction between bus charge and battery swap stations for taxi and bus fleets. [42] propose a mixed-integer second-order cone programming model for the charging planning of battery-electric buses.

2.3 Contributions

1. **Modeling**: This thesis proposes an Optimal Planning of Charging Facilities and Electric Bus Fleet (OPCF-EBF) model for public transit systems to optimally plan the transition to an entirely electric bus fleet with several innovative features.

   (a) The two-time-scale structure of the OPCF-EBF model brings together long-term planning and short-term operation, with annual investment decisions of charging infrastructure and fleet composition over a decade-long transition horizon and hourly operation decisions of bus charging and scheduling over 24 hours in each planning year.

   (b) Modular arithmetic is used to model the repeating of daily 24-hour bus demand in each planning year. Various charging strategies, such as early charging (charging before battery full depletion), idling (neither working nor charging), non-preemptive charging (charging until full), are modeled, together with practical planning strategies such as utilization of existing bus depots and route terminals as potential charging facilities, respecting of retirement schedules of conventional buses, and various realistic investment and operational costs.

2. **Characterizations**: We characterize the computational complexity of the proposed model and identify special structures in a subclass of the proposed model that can be

polynomially solved through an interesting transformation.

(a) The proposed OPCF-EBF model is shown to be NP-hard through reduction from the uncapacitated facility location problem. Even with one planning period and two charging states for depot-charged BEBs, the OPCF-EBF model is still NP-hard, as the numbers of bus routes and charging depots grow.

(b) We show in Chapter 3 that an important class of OPCF-EBF problems, which has one bus route and an arbitrary number of charging states for depot-charged BEBs under a simple charging policy, has a nice constraint structure after a linear transformation and is polynomially solvable.

3. **Algorithm**: We propose an effective and computationally scalable primal heuristic called "Policy-Restriction" that significantly outperforms and improves Gurobi.

4. **Real-world case studies**: We conduct extensive computational studies using real-world data from public transit systems in major cities in the U.S. and around the world, which reveal insights into the optimal investment and operational strategies. For example, an optimal investment decision tends to invest in depot chargers before on-route chargers; an optimal operational solution tends to use on-route BEBs to meet base-load bus demand and to use depot BEBs to meet peaking bus demand. These empirical insights are also explained through theoretical analysis.

The rest of Chapter 2 is structured as follows. We introduce the OPCF-EBF model in Section 2.4. In Section 2.5, we analyze the complexity of the proposed OPCF-EBF model. Section 2.6 proposes a primal heuristic to solve the challenging large-scale integer optimization model. Section 2.7 reports real-world case studies with observed insights and theoretical analysis. Section 2.8 concludes Chapter 2.

2.4 An optimal planning model for charging facility and battery electric bus fleet

The OPCF-EBF model is formulated as a two-stage problem in Section 2.4.1, where investment problem is in the first stage and the operational problem is the second-stage recourse. The detailed operational problem is formulated in Section 2.4.2.

2.4.1 The OPCF-EBF model

We first define all the investment parameters and investment decision variables before laying out the overall two-stage OPCF-EBF model.

Investment parameters

Let Θ denote the set of yearly investment periods, (i.e., $\Theta = \{1, 2 \ldots, N\}$ for some $N \in \mathbb{Z}_+$). Let $\mathcal{I}$ be the set of depot sites, $\mathcal{J}$ be the set of bus routes, $\mathcal{K}$ be the set of relevant depot chargers, and $\mathcal{R}$ be the set of terminal stations available for on-route charging. In the model, we allow BEBs from different manufacturers, since different BEB models can have different charging times, battery capacities, unit costs, and ability to perform on-route charging. In particular, denote $\mathcal{B}_{depot}$ and $\mathcal{B}_{route}$ as the set of BEB models that can be charged by depot and on-route chargers, respectively.

Investment decision variables

Let $x \in \{0, 1\}^{|\mathcal{I}| \times |\Theta|}$ be the vector of binary variables such that $x_i^\theta = 1$ implies that the depot site $i \in \mathcal{I}$ can install depot chargers during the investment period $\theta \in \Theta$, and $x_i^\theta = 0$ otherwise. Let $y \in \mathbb{Z}_+^{|\mathcal{I}| \times |\mathcal{K}| \times |\Theta|}$ be the vector of integer variables whose entry y_{ik}^θ represents the number of depot chargers of a given plug type $k \in \mathcal{K}$ (e.g. levels 1 and 2 chargers) for a given site $i \in \mathcal{I}$ and investment period $\theta \in \Theta$. Let $\chi \in \mathbb{Z}_+^{|\mathcal{R}| \times |\Theta|}$ represent the number of on-route chargers at each terminal station $r \in \mathcal{R}$ and investment period $\theta \in \Theta$. Let $\eta \in \mathbb{Z}_+^{|\mathcal{B}_{depot}| \times |\mathcal{J}| \times |\Theta|}$ and $\widetilde{\eta} \in \mathbb{Z}_+^{|\mathcal{B}_{route}| \times |\mathcal{J}| \times |\Theta|}$ be the numbers of depot and on-route

BEBs respectively along each route $j \in \mathcal{J}$, given the BEB type, and investment period. Let $\xi \in \mathbb{Z}_+^{|\mathcal{J}|\times|\Theta|}$ be the vector of conventional buses in each route $j \in \mathcal{J}$ and investment period $\theta \in \Theta$.

Two-stage OPCF-EBF model

The OPCF-EBF model is formulated as a two-stage integer optimization model in the investment variables below.

$$\min \quad \sum_{\theta\in\Theta} \gamma_\theta \cdot \Big(I_\theta(x, y, \chi, \eta, \widetilde{\eta}, \xi) + F_\theta(x, y, \chi, \eta, \widetilde{\eta}, \xi) \Big) \tag{2.1a}$$

$$\text{s.t.} \quad I_\theta(x, y, \chi, \eta, \widetilde{\eta}, \xi) \le C_\theta, \qquad \theta \in \Theta, \tag{2.1b}$$

$$x_i^\theta \ge x_i^{\theta-1}, \quad y_{ik}^\theta \ge y_{ik}^{\theta-1}, \quad \chi_r^\theta \ge \chi_r^{\theta-1}, \qquad i \in \mathcal{I}, \quad k \in \mathcal{K}, \quad r \in \mathcal{R}, \quad \theta \in \Theta, \tag{2.1c}$$

$$\eta_{bj}^\theta \ge \eta_{bj}^{\theta-1}, \quad \widetilde{\eta}_{\bar{b}j}^\theta \ge \widetilde{\eta}_{\bar{b}j}^{\theta-1}, \quad \xi_j^\theta \le \xi_j^{\theta-1}, \qquad b \in \mathcal{B}_{depot}, \quad \bar{b} \in \mathcal{B}_{route}, \quad j \in \mathcal{J}, \quad \theta \in \Theta, \tag{2.1d}$$

$$\underline{Q}_{ik}^\theta x_i^\theta \le y_{ik}^\theta \le \overline{Q}_{ik}^\theta x_i^\theta, \quad 0 \le \chi_r^\theta \le \chi_{UB,r}^\theta, \qquad i \in \mathcal{I}, \quad k \in \mathcal{K}, \quad r \in \mathcal{R}, \quad \theta \in \Theta, \tag{2.1e}$$

$$\xi_{LB,j}^\theta \le \xi_j^\theta \le \xi_{UB,j}^\theta, \qquad j \in \mathcal{J}, \quad \theta \in \Theta, \tag{2.1f}$$

$$x_i^\theta \in \{0,1\}, \; y_{ik}^\theta \in \mathbb{Z}_+, \; \chi_r^\theta \in \mathbb{Z}_+, \qquad i \in \mathcal{I}, \quad k \in \mathcal{K}, \quad r \in \mathcal{R}, \quad \theta \in \Theta, \tag{2.1g}$$

$$\eta_{bj}^\theta \in \mathbb{Z}_+, \; \widetilde{\eta}_{\bar{b}j}^\theta \in \mathbb{Z}_+, \; \xi_j^\theta \in \mathbb{Z}_+, \qquad b \in \mathcal{B}_{depot}, \quad \bar{b} \in \mathcal{B}_{route}, \quad j \in \mathcal{J}, \quad \theta \in \Theta. \tag{2.1h}$$

The objective function (2.1a) has two parts I_θ and F_θ for each investment period θ, where I_θ represents the investment related cost, F_θ is the operational cost associated with the infrastructure decision $(x, y, \chi, \eta, \widetilde{\eta}, \xi)$, and γ_θ is a discount factor. The investment cost I_θ is defined as

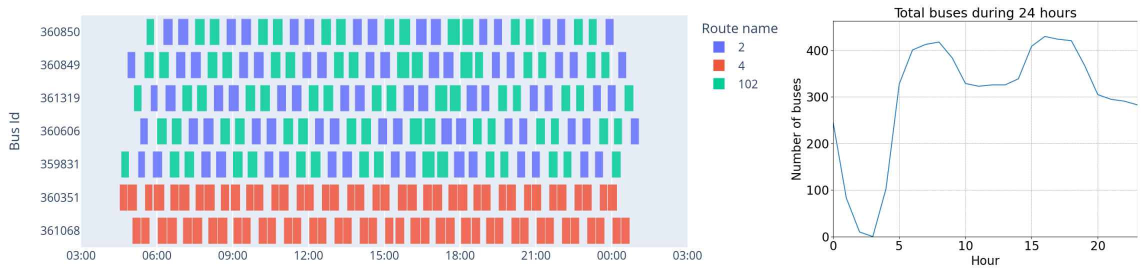
$$
\begin{aligned}
I_\theta(x,\chi,y,\eta,\widetilde{\eta},\xi) = &\sum_{i\in\mathcal{I}} c_{x,i}^\theta (x_i^\theta - x_i^{\theta-1}) + \sum_{(i,k)\in\mathcal{I}\times\mathcal{K}} c_{y,ik}^\theta (y_{ik}^\theta - y_{ik}^{\theta-1}) \\
&+ \sum_{r\in\mathcal{R}} c_{\chi,r}^\theta (\chi_r^\theta - \chi_r^{\theta-1}) + \sum_{(b,j)\in\mathcal{B}_{depot}\times\mathcal{J}} c_{\eta,bj}^\theta (\eta_{bj}^\theta - \eta_{bj}^{\theta-1}) \\
&+ \sum_{(b,j)\in\mathcal{B}_{depot}\times\mathcal{J}} c_{\widetilde{\eta},bj}^\theta (\widetilde{\eta}_{bj}^\theta - \widetilde{\eta}_{bj}^{\theta-1}) + \sum_{j\in\mathcal{J}} c_{\xi,j}^\theta \cdot (\xi_j^\theta - \xi_j^{\theta-1}),
\end{aligned}
\tag{2.2}
$$

which is incurred on the incremental change of charging facilities and bus fleet in year θ. The vectors c_x, c_y, c_χ, c_η, $c_{\widetilde{\eta}}$, and c_ξ in (2.2) represent the unit cost of the corresponding decisions x, y, χ, η, $\widetilde{\eta}$, and ξ. The cost c_ξ signifies the financial benefit of retiring a conventional bus, which can be also interpreted as a penalty for using fossil fuel-based buses. The operational cost F_θ is given by a recourse problem in operational decisions and will be presented in Section 2.4.2.

In terms of constraints, we have the budget constraint (2.1b) on the investment costs, where C_θ is the investment budget in period θ. Constraints (2.1c) and (2.1d) describe the monotone expansion of the charging infrastructure and the BEB fleet, and the monotone reduction of the conventional bus fleet. We have the upper and lower bounds (2.1e) on the numbers of depot and on-route chargers, where $\underline{Q}_{ik}^\theta$ and $\overline{Q}_{ik}^\theta$ are the upper and lower bounds on the number of depot plugs y_{ik}^θ given that the depot site i is open (i.e., $x_i^\theta = 1$), while $\chi_{UB,r}^\theta$ is the upper bound on the number of on-route chargers χ_r^θ. The bus retirement target constraint (2.1f) has upper and lower bounds $\xi_{LB,j}^\theta$ and $\xi_{UB,j}^\theta$ for the number of conventional buses ξ_j^θ in each year θ.

2.4.2 The operational problem as the recourse

The goal of the operational problem is to find an optimal daily schedule for the charging and operation of a mixed fleet of BEBs and conventional buses in an investment period with a given investment decision. We use the information on existing bus routes and schedules published by public transit agencies, see [43], and assume that the mixed fleet should operate on the same routes and satisfy the same bus demand as in the current system. To obtain the bus demand for each hour and route, we count the number of operating buses from the published bus schedules. See Figure 2.1a as an illustration of the bus schedules for routes 2, 4, and 102 on a weekday in Atlanta's MARTA system and Figure 2.1b for the total bus demand in Atlanta on a weekday in August 2019.



(a) Weekday bus schedule for routes 2, 4, and 102.

(b) Weekday total bus demand.

Figure 2.1: Atlanta MARTA bus schedule for routes 2,4, and 102, and the total bus demand in August 2019.

In the following sections, we describe the operational problem as a recourse to the investment decisions. Sections 2.4.2, 2.4.2, and 2.4.2 describe the parameters, decisions, and constraints, which are put together in Section 2.4.2 to formulate the operational problem, whose optimal objective value is the operational cost F_θ in the overall model (2.1a).

Parameters in the operational problem.

1. **Time related parameters:** Let $[0 : T-1]$ be the set of time intervals $\{0, 1, \ldots, T-1\}$ of the operational horizon, which is treated as the cyclic group $\mathbb{Z}/T\mathbb{Z}$ of integers modulo T. In this way, the operation becomes cyclic, i.e., the operation at time $t = T-1$ loops back to time $t = 0$ as the next time step. This construction models stationary, periodic operation rather than transient operation with fixed initial ($t = 0$) and final conditions ($t = T-1$). If each $t \in [0 : T-1]$ is an hourly interval and $T = 24$, then the operational problem models the repeated daily bus operation.

2. **Charging related parameters:** Different types of BEBs may have different battery

capacities, thus we define $[1 : W_b + 1] := \{1, \dots, W_b + 1\}$ as the set of battery states of a depot BEB of type $b \in \mathcal{B}_{depot}$, where $s = 1$ and $s = W_b + 1$ denote the fully charged and the fully discharged states, respectively. The value W_b is the lowest battery state in which it is still safe to operate a depot BEB. The battery state index increases with time, that is, if a depot BEB is in a state s when it operates in time interval t, then it must be in the battery state $s + 1$ at time $t + 1$ to model the battery discharging in one interval of operation. Let $\mathcal{P}$ be the set of all pairs of depot BEB types and battery states, i.e., $\mathcal{P} := \{(b, s) : b \in \mathcal{B}_{depot},\ s \in [1 : W_b + 1]\}$. Let L_{bks} be the number of time intervals needed to fully charge a depot BEB of type $b \in \mathcal{B}_{depot}$ from state $s \in [1 : W_b + 1]$ to $s = 1$ using a depot charger of type $k \in \mathcal{K}$. For instance, suppose an hourly interval operational model with $T = 24$ hours. Consider a depot BEB b with battery capacity $W_b = 12$ hours. If a depot charger k takes 6 hours to fully charge b, then we have $L_{bk,13} = 6$. One can use a linear interpolation rule to define the charging time L_{bks} for other states of charges $s \in [1 : W_b]$. This means that if the BEB b starts to charge at time t with initial state $s = 13$, then it will be fully charged at time $t + 6$.

3. **Routes related parameters:** Let $\mathcal{J}(r)$ be the set of bus routes associated with terminal station r. Let $\mathcal{R}(j)$ be the set of terminal stations that are connected to the bus route j. We assume it is possible to accommodate up to CH_r on-route charging activities within one operational time interval at the terminal station r. Denote by $\mathcal{Q}$ the set of all pairs of routes and terminal stations, i.e., $\mathcal{Q} := \{(j, r) : j \in \mathcal{J},\ r \in \mathcal{R}(j)\}$.

Decision variables in the operational problem.

Given an investment period $\theta \in \Theta$, consider the decisions $w^\theta \in \mathbb{Z}_+^{\mathcal{N}_{d,op}}$ and $v^\theta \in \mathbb{Z}_+^{\mathcal{N}_{d,op}}$ as the number of depot BEBs that are working and idling (i.e. neither working nor charging) respectively, where the set $\mathcal{N}_{d,op}$ is the Cartesian product $\mathcal{P} \times \mathcal{J} \times [0 : T - 1]$ and represents all the indices for w^θ and v^θ, including the depot BEB model, battery state, bus route, and

time interval. Moreover, let $z \in \mathbb{Z}_+^{\mathcal{N}_{d,ch}}$ represent the battery state at which a group of depot BEBs starts charging, where $\mathcal{N}_{d,ch}$ is defined as $\mathcal{P} \times \mathcal{I} \times \mathcal{J} \times \mathcal{K} \times [0 : T-1]$. Finally, the decision $\beta^\theta \in \mathbb{Z}_+^{\mathcal{N}_{d,beta}}$ contains the number of depot BEBs that are currently charging regardless of the charging state, where $\mathcal{N}_{d,beta}$ is defined as $\mathcal{B} \times \mathcal{J} \times [0 : T-1]$.
Let $\widetilde{w}^\theta \in \mathbb{Z}_+^{\mathcal{N}_{route}}$ and $\widetilde{v}^\theta \in \mathbb{Z}_+^{\mathcal{N}_{route}}$ be the decisions that represent the number of on-route BEBs that are working and idling respectively, for each terminal station, route, and time interval. The set $\mathcal{N}_{route}$ is defined as $\mathcal{Q} \times [0 : T-1]$. Let $\phi^\theta \in \mathbb{Z}_+^{\mathcal{N}_{conv}}$ and $\sigma^\theta \in \mathbb{Z}_+^{\mathcal{N}_{conv}}$ be the numbers of conventional buses working and idling for each route and time interval, where $\mathcal{N}_{conv}$ is $\mathcal{J} \times [0 : T-1]$. Finally, let $u^\theta \in \mathbb{Z}_+^{\mathcal{N}_{slack}}$ be the slack variable of the demand constraint, where $\mathcal{N}_{slack}$ is $\mathcal{J} \times [0 : T-1]$.

Constraints of the operational problem.

1. **Bus demand satisfaction:** Given $(j, t, \theta) \in \mathcal{J} \times [0 : T-1] \times \Theta$, we have the demand constraint

$$\sum_{(b,s)\in\mathcal{P}} w_{bjs}^{t,\theta} + \sum_{(b,r)\in\mathcal{B}_{route}\times\mathcal{R}} \widetilde{w}_{bjr}^{t,\theta} + \phi_j^{t,\theta} + u_j^{t,\theta} \geq d_j^{t,\theta}, \tag{2.3}$$

   where the bus demand $d_j^{t,\theta}$ must be satisfied by the total number of working depot BEBs (the first term on the left), working on-route BEBs (the second term), working conventional buses (the third term), and the slack variable $u_j^{t,\theta}$ (the fourth term).

2. **Depot BEB dynamics:** For all $t \in [0 : T-1]$ and $(b, j, \theta) \in \mathcal{B}_{depot} \times \mathcal{J} \times \Theta$, we

have the depot BEB charging dynamic equations

$$w_{bj1}^{t,\theta} + v_{bj1}^{t,\theta} = \sum_{s=2}^{W_b+1} \sum_{(i,k)\in\mathcal{I}\times\mathcal{K}} z_{bijks}^{(t-L_{bks}),\theta} + v_{bj1}^{(t-1),\theta}, \tag{2.4a}$$

$$w_{bjs}^{t,\theta} + \sum_{(i,k)\in\mathcal{I}\times\mathcal{K}} z_{bijks}^{t,\theta} + v_{bjs}^{t,\theta} = w_{bj(s-1)}^{(t-1),\theta} + v_{bjs}^{(t-1),\theta}, \qquad s \in [2:(W_b+1)], \tag{2.4b}$$

$$w_{bj(W_b+1)}^{t,\theta} = 0, \quad z_{bijk1}^{t,\theta} = 0. \tag{2.4c}$$

Equation 2.4a states that the total number of fully charged ($s = 1$), working and idling depot BEBs at time t (the two terms on the left) must equal to the total number of depot BEBs that just finished charging at time t (the first term on the right) plus the fully charged idle depot BEBs at time $t-1$ (the second term on the right). The same dynamics applies to the partially charged state $s \in [2 : (W_b + 1)]$ in (2.4b). Equation (2.4c) enforces that the depot BEBs cannot work if fully depleted (the first equation) and cannot charge if fully charged (the second equation). Recall that all time indices are cyclic modulo T. Also note that these equations represent a non-preemptive charging policy (i.e. charging must continue until fully charged), which is practical for depot BEB charging and assumed throughout the paper.

3. **On-route BEB and conventional bus dynamics:** The dynamic equations for on-route BEBs and conventional buses are

$$\widetilde{w}_{bjr}^{t,\theta} + \widetilde{v}_{bjr}^{t,\theta} = \widetilde{w}_{bjr}^{(t-1),\theta} + \widetilde{v}_{bjr}^{(t-1),\theta}, \tag{2.5a}$$

$$\phi_{j}^{t,\theta} + \sigma_{j}^{t,\theta} = \phi_{j}^{(t-1),\theta} + \sigma_{j}^{(t-1),\theta}, \tag{2.5b}$$

for all $(j,\theta) \in \mathcal{J} \times \Theta$, $r \in \mathcal{R}(j)$, $b \in \mathcal{B}_{route}$, and $t \in [0 : T-1]$. Equations (2.5a) and (2.5b) are conservation of on-route and conventional buses over each time interval, respectively. Because the on-route charging is accommodated within an operational time interval no state-of-charge index is needed.

4. **Bounds on depot BEBs simultaneously being charged:** The number of depot BEBs being charged and the upper bound by the number of depot chargers are given below

$$\beta_{bijk}^{t,\theta} - \sum_{s=2}^{W_b+1} \sum_{l=0}^{L_{bks}-1} z_{bijks}^{(t-l),\theta} = 0, \tag{2.6a}$$

$$\sum_{b \in \mathcal{B}_{depot}} \sum_{j \in \mathcal{J}} \beta_{bijk}^{t,\theta} \leq y_{ik}^{\theta}, \tag{2.6b}$$

for all $(i, j, k, \theta) \in \mathcal{I} \times \mathcal{J} \times \mathcal{K} \times \Theta$, $b \in \mathcal{B}_{depot}$, and $t \in [0 : T-1]$. Here, equation (2.6a) has the total number of type b depot BEBs that are being charged at time t, depot i, route j, using charging plug type k. Equation (2.6b) is an upper bound on $\beta_{bijk}^{t,\theta}$ by the total number of depot chargers y_{ik}^{θ} that are invested.

5. **On-route charging capacity:** The number of working on-route BEBs that can charge at a given terminal is bounded by the following constraint

$$\sum_{b \in \mathcal{B}_{route}} \sum_{j \in \mathcal{J}(r)} \widetilde{w}_{bjr}^{t,\theta} \leq CH_r \cdot \chi_r^{\theta}, \tag{2.7}$$

for all $(r, \theta) \in \mathcal{R} \times \Theta$, and $t \in [0 : T-1]$. Recall that CH_r is the charging capacity of an on-route charger over a time interval and χ_r^{θ} is the number of on-route chargers on route r, investment period θ.

6. **Total numbers of BEBs and conventional buses:** The link between the operational variables and the total number of BEBs and conventional buses is given below

$$\sum_{s=1}^{W_b} w_{bjs}^{0,\theta} + \sum_{s=1}^{W_b+1} v_{bjs}^{0,\theta} + \sum_{i \in I} \sum_{k \in K} \beta_{bijk}^{0,\theta} = \eta_{bj}^{\theta}, \quad b \in \mathcal{B}_{depot}, \tag{2.8a}$$

$$\sum_{r \in \mathcal{R}(j)} \widetilde{w}_{bjr}^{0,\theta} + \widetilde{v}_{bjr}^{0,\theta} = \widetilde{\eta}_{bj}^{\theta}, \quad b \in \mathcal{B}_{route}, \tag{2.8b}$$

$$\phi_j^{0,\theta} + \sigma_j^{0,\theta} = \xi_j^{\theta}, \tag{2.8c}$$

for all $(j,\theta) \in \mathcal{J} \times \Theta$. It is enough to relate the total number of buses to the operational variables at time $t = 0$, because the dynamic equations (2.4)-(2.5) imply bus conservation, see Section 2.4.3.

The model of the operational problem

Finally, we can formulate the operational problem using the constraints defined above:

$$\begin{aligned} F_\theta(x, y, \chi, \eta, \widetilde{\eta}, \xi) := \min \quad & p_z^\top z^\theta + p_w^\top w^\theta + p_{\tilde{w}}^\top \widetilde{w}^\theta + p_\phi^\top \phi^\theta + p_u^\top u^\theta \\ \text{s.t.} \quad & (2.3) - (2.8), \\ & w^\theta, v^\theta \in \mathbb{Z}_+^{\mathcal{N}_{d,op}},\ z^\theta \in \mathbb{Z}_+^{\mathcal{N}_{d,ch}},\ \beta^\theta \in \mathbb{Z}_+^{\mathcal{N}_{d,beta}}, \\ & \widetilde{w}^\theta, \widetilde{v}^\theta \in \mathbb{Z}_+^{\mathcal{N}_{route}},\ \phi^\theta, \sigma^\theta \in \mathbb{Z}_+^{\mathcal{N}_{conv}}, u^\theta \in \mathbb{Z}_+^{\mathcal{N}_{slack}}. \end{aligned} \tag{2.9}$$

Some observations are instructive regarding the operational costs. The cost p_z contains the unit electricity cost for charging a depot BEB plus the deadhead cost of a trip between a route and a depot charging site. The p_w and p_ϕ represent the unit costs of operating depot BEBs and conventional buses, respectively, which are essentially the bus driver costs. The $p_{\tilde{w}}$ contains the unit electricity cost associated with the incremental charge at on-route stations plus the bus driver cost. The p_u represents the penalty for the demand constraint violation.

2.4.3 Properties of the optimal planning of model

Conservation of the total number of buses

As our model does not track individual buses, but rather only tracks the total number of buses in different states, it would be assuring to verify that the total number of each type of buses in the fleet is conserved over operating times within each investment period. Indeed, Eq. (2.5a) and (2.8b) imply that the total number of on-route BEBs counted in interval t is equal to the total number of invested on-route BEBs $\widetilde{\eta}_{bj}^{\theta}$ as

$$\sum_{r \in \mathcal{R}(j)} \widetilde{w}_{bjr}^{t,\theta} + \widetilde{v}_{bjr}^{t,\theta} = \widetilde{\eta}_{bj}^{\theta}, \qquad b \in \mathcal{B}_{route}, \tag{2.10}$$

for all $(j, \theta) \in \mathcal{J} \times \Theta$ and $t \in [0 : T - 1]$. The conservation of the total number of conventional buses follows analogously as $\phi_j^{t,\theta} + \sigma_j^{t,\theta} = \xi_j^{\theta}$ for all $j \in \mathcal{J}$, $t \in [0 : T-1]$, and $\theta \in \Theta$. For depot BEBs, the conservation is stated in Lemma 2.4.1.

Lemma 2.4.1. *The total number of depot BEBs is constant through the operational horizon. That is, the following equality holds*

$$\sum_{s=1}^{W_b} w_{bjs}^{t,\theta} + \sum_{s=1}^{W_b+1} v_{bjs}^{t,\theta} + \sum_{i \in I} \sum_{k \in K} \beta_{bijk}^{t,\theta} = \eta_{bj}^{\theta}, \qquad \begin{array}{ll} t \in [0 : T-1], & b \in \mathcal{B}_{depot}, \\ j \in \mathcal{J}, & \theta \in \Theta. \end{array} \tag{2.11}$$

Proof of Lemma 2.4.1. Equation (2.11) is proved by induction. Indeed, the base case $t = 0$ follows from the constraint (2.8a).Denote by C_t the term $\sum_{s=1}^{W_b} w_{bjs}^{t,\theta} + \sum_{s=1}^{W_b+1} v_{bjs}^{t,\theta}$. Note that on the left-hand side of (2.11) we have

$$\begin{aligned} \sum_{s=1}^{W_b} w_{bjs}^{t,\theta} + \sum_{s=1}^{W_b+1} v_{bjs}^{t,\theta} + \sum_{i \in I} \sum_{k \in K} \beta_{bijk}^{t,\theta} &= C_t + \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} \sum_{s=2}^{W_b+1} \sum_{l=0}^{L_{bks}-1} z_{ijks}^{(t-l),\theta} \\ &= C_t + \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} \sum_{s=2}^{W_b+1} \left(\sum_{l=0}^{L_{bks}-2} z_{bijks}^{(t-l-1),\theta} + z_{bijks}^{t,\theta} \right), \end{aligned} \tag{2.12}$$

where equation (2.12) is obtained by splitting the sum $\sum_{l=0}^{L_{ks}-1} z_{bijks}^{(t-l),\theta}$ into the summation of $\sum_{l=1}^{L_{ks}-1} z_{ijks}^{(t-l),\theta}$ and $z_{ijkst}^{t,\theta}$, and then by re-indexing l from 0 to $L_{bks} - 2$. Sum the depot transition equations (2.4a) and (2.4b) over the states of charge $s \in [1 : W_b + 1]$ to get

$$C_t + \sum_{i \in I} \sum_{k \in K} \sum_{s=2}^{W_b+1} z_{ijks}^{t,\theta} = C_{(t-1)} + \sum_{i \in I} \sum_{k \in K} \sum_{s=2}^{W_b+1} z_{ijks}^{(t-L_{ks}),\theta}. \tag{2.13}$$

Replace (2.13) into (2.12) and use the induction hypothesis for $t-1$ to conclude the proof:

$$\begin{aligned}
&\sum_{s\in[1:W]} w_{bjs}^{t,\theta} + \sum_{s=1}^{W_b+1} v_{bst}^{t,\theta} + \sum_{i\in I}\sum_{k\in K}\beta_{bijk}^{t,\theta} \\
&\overset{(2.12)+(2.13)}{=} C_{(t-1)} + \sum_{i\in\mathcal{I}}\sum_{k\in\mathcal{K}}\sum_{s=2}^{W_b+1}\left(\sum_{l=0}^{L_{ks}-2} z_{ijks}^{(t-l-1),\theta} + z_{ijks}^{(t-L_{ks}),\theta}\right) \\
&= C_{(t-1)} + \sum_{i\in\mathcal{I}}\sum_{k\in\mathcal{K}}\sum_{s=2}^{W_b+1}\sum_{l=0}^{L_{bks}-1} z_{ijks}^{(t-1-l),\theta} \overset{(\beta\text{ def.})}{=} C_{(t-1)} + \sum_{i\in I}\sum_{k\in K}\beta_{bijk}^{(t-1),\theta} = \eta_{bj}^{\theta}.
\end{aligned}$$

□

2.5 Computational complexity

2.5.1 Complexity of the OPCF-EBF model

The OPCF-EBF problem defined in (2.1)-(2.9) is NP-hard. The idea of the proof is to create a mapping between the charging depot and bus terminals in the OPCP-EBF to the facilities and customers in the uncapacitated facility location problem. Moreover, it turns out that some special classes of the OPCF-EBF problem are already NP-hard as shown in the following theorem.

Theorem 2.5.1. *The OPCF-EBF problem is NP-hard. In particular, even an OPCF-EBF problem with a single investment period and only depot BEBs of two battery states or only on-route BEBs is NP-hard.*

Proof of Theorem 2.5.1. Let us denote by $\lambda_i \in \{0,1\}$ the binary variable that corresponds to decision to open or not the facility $i \in [n] := \{1,\ldots,n\}$ and by $\pi_{ij} \in \{0,1\}$ the binary variable that corresponds to meet the demand of j-th client using the i-th installation. Consider the facility setup cost f_i associated with variable λ_i and the supply cost g_{ij} associated with λ_{ij}. Below, we present an instance of the UFL problem:

$$\begin{array}{lll} \min & \sum_{i=1}^{n} f_i\lambda_i + \sum_{i=1}^{n}\sum_{j=1}^{m} g_{ij}\pi_{ij} & \\ \text{s.t.} & \sum_{i=1}^{n} \pi_{ij} = 1, & \forall j \in [m], \\ & \sum_{j=1}^{m} \pi_{ij} \leq m \cdot \lambda_i, & \forall i \in [n], \\ & \lambda_i \in \{0,1\},\ \pi_{ij} \in \{0,1\}, & \forall i \in [n],\ \forall j \in [m]. \end{array}$$

We now define the reduction to an instance of the OPCF-EBF problem starting with the set of indices. Consider just one type of depot BEB, $\mathcal{B}_{depot} = \{1\}$, but not a single on-route BEB, $\mathcal{B}_{route} = \emptyset$, n potential charging sites, $\mathcal{I} = \{1, \ldots, n\}$, m routes, $\mathcal{J} = \{1, \ldots, m\}$, only one plug type, $\mathcal{K} = \{1\}$, not a single on-route charging facility, $\mathcal{R} = \emptyset$, battery performance of one time-interval, $W_1 = 1$, operational horizon $T = 2$, and single investment period $\Theta = \{1\}$. To improve the presentation of this instance of the OPCF-EBF problem, we omit the sub-indices that have only one possible value such as the depot BEB type b, the plug type k, and the investment period θ.

Consider the lower $\underline{Q}_i$ and upper $\overline{Q}_i$ bounds of plugs as being equal to 0 and m, respectively, for all site $i \in \mathcal{I}$. Let d_{jt} be the demand for buses and let L_s be the charging time as defined below

$$d_{jt} = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{if } t = 1, \end{cases}, \qquad L_s = \begin{cases} 0, & \text{if } s = 1, \\ 1, & \text{if } s = 2, \end{cases}$$

for every route $j \in \mathcal{J}$. Note that $s = 1$ is the fully charged state, and $s = 2$ is the fully depleted state, since $W = 1$. The idea of our construction is to have the depot BEBs working at time $t = 0$ and charging at time $t = 1$.

We assume that the initial infrastructure condition is zero, that is, $x_{i0} = 0$, $y_{i0} = 0$, $\eta_{j0} = 0$, and $\xi_{j0} = 0$, for all depots $i \in [n]$ and routes $j \in [m]$. If the initial condition of conventional buses is zero, $\xi_{j0} = 0$, then the number of conventional buses in the period of investment $\theta = 1$ is also zero, that is, $\xi_{j1} = 0$. This implies that the number of working $\phi_{jt\theta}$ and idling $\sigma_{jt\theta}$ conventional buses are zero for all time intervals $t \in [0 : 1]$, route $j \in [m]$, and investment period $\theta = 1$.

Let H be the constant $\sum_{i=1}^{n} f_i + \sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij}$, and consider the unit cost of a depot BEB c_{beb} as $H + 1$. Let the investment budget C be equal to $\sum_{i=1}^{n} f_i + m \cdot c_{beb}$, which is essentially a large enough constant so all possible investments are feasible. Then the investment part of this OPCF-EBF instance is given below:

$$
\begin{aligned}
\min \quad & \textstyle\sum_{i=1}^{n} f_i x_i + \sum_{j=1}^{m} c_{beb} \cdot \eta_j + F(x, y, \eta) \\
\text{s.t.} \quad & \textstyle\sum_{i=1}^{n} f_i x_i + \sum_{j=1}^{m} c_{beb} \cdot \eta_j \leq C, \\
& 0 \leq y_i \leq m \cdot x_i, && \forall i \in [n], \\
& x_i \in \{0, 1\},\ y_i,\ \eta_j \in \mathbb{Z}_+, && \forall i \in [n],\ \forall j \in [m],
\end{aligned}
$$

and there is no on-route and conventional bus variables since those are zero.

For the operational problem, we have two remarks regarding the depot working and charging variables w and z, respectively. We omit the state of charge $s = 1$ of the depot working variable w, since this is the only possible state for a working depot BEB given that $W = 1$. Similarly, we omit the state of charge $s = 2$ for z, since this is also the only possible state of charge for a depot BEB to start charging in our instance. For the depot idling BEBs v, we keep the state of charge index $s \in [1 : 2]$ because it is possible for a depot BEB to be idle in both fully charged ($s = 1$) and fully discharged ($s = 2$) states. Let the demand constraint violation cost c_u be equal to $(m + 1) \cdot c_{beb}$. Below, we present the operational part of our OPCF-EBF instance:

$$
\begin{aligned}
F(x,y,\eta) = \min \quad & \textstyle\sum_{i=1}^{n}\sum_{j=1}^{m}\left[0\cdot z_{ij}^{0} + g_{ij}\cdot z_{ij}^{1}\right] + \sum_{j=1}^{m}\sum_{t=0}^{1} c_u \cdot u_{jt} \\
\text{s.t.} \quad & w_j^t + u_j^t \geq d_j^t, && \forall j\in[m],\ \forall t\in[0:1] \\
& w_j^t + v_{j1}^t = \textstyle\sum_{i=1}^{n} z_{ij}^{(t-1)} + v_{j1}^{(t-1)}, && \forall j\in[m],\ \forall t\in[0:1], \\
& \textstyle\sum_{i=1}^{n} z_{ij}^t + v_{j2}^t = w_j^{(t-1)} + v_{j2}^{(t-1)}, && \forall j\in[m],\ \forall t\in[0:1], \\
& \beta_{ij}^t = z_{ij}^t, && \forall i\in[n],\ \forall j\in[m],\ \forall t\in[0:1], \\
& \textstyle\sum_{j\in[m]} \beta_{ij}^t \leq y_i, && \forall i\in[n],\ \forall t\in[0:1], \\
& \eta_j = \left(v_{j1}^t + v_{j2}^t + w_j^t + \textstyle\sum_{i=1}^{n}\beta_{ij}^t\right), && \forall j\in[m],\ \forall t=0, \\
& w_j^t,\ v_{js}^t,\ z_{ij}^t,\ \beta_{ij}^t,\ u_j^t \in \mathbb{Z}_+, && \forall i\in[n],\ \forall j\in[m],\ \forall t\in[0:1], \\
& \forall s\in[1:2].
\end{aligned}
$$

Now that we have defined the instance of the OPCF-EBF, we focus on the reduction of the UFL problem. Let (λ, π) be a feasible solution of the UFL problem with an objective value less than or to K. Note that K is less than or equal to $H = \sum_{i=1}^{n} f_i + \sum_{i=1}^{n}\sum_{j=1}^{m} g_{ij}$, because H is an upper bound for the UFL objective cost. Consider the following OPCF-EBF-induced solution:

$$x_i = \xi_i, \quad y_i = m\cdot\lambda_i, \quad \eta_j = 1, \tag{2.14a}$$

$$w_j^t = \begin{cases} 1, & \text{if } t=0, \\ 0, & \text{if } t=1, \end{cases} \qquad z_{ij}^t = \begin{cases} 0, & \text{if } t=0, \\ \pi_{ij}, & \text{if } t=1, \end{cases}, \tag{2.14b}$$

$$v_{js}^t = 0, \quad \beta_{ij}^t = z_{ij}^t, \quad u_j^t = 0, \tag{2.14c}$$

for every site $i \in [n]$, route $j \in [m]$, charge state $s \in [1:2]$, and time interval $t \in [0:1]$. The solution defined by the equations (2.14a), (2.14b), and (2.14c) is feasible for the OPCF-EBF instance, and it has objective value equal to $\sum_{i=1}^{n} f_i\lambda_i + \sum_{i=1}^{n}\sum_{j=1}^{m} g_{ij}\pi_{ij} + c_{beb}\cdot m$, which is equal to the objective value of the UFL problem plus the constant $c_{beb}\cdot m$. Therefore, the objective value of the OPCF-EBF instance is less or equal to $K + c_{beb}\cdot m$, which is also less than or equal to $H + c_{beb} \cdot m$.

We now check the other side of the reduction. Consider a feasible solution of the OPCF-EBF instance $(x, y, \eta, w, v, z, \beta, u)$ with objective value $K + c_{beb} \cdot m$, where K is less than or equal to H. Such solution exist, since we can take $(x, y, \eta, w, v, z, \beta, u)$ as defined by (2.14a), (2.14b), and (2.14c), and the following feasible solution (λ, π) for the UFL problem:

$$\lambda_i = \begin{cases} 1, & \text{if } i = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad \pi_{ij} = \begin{cases} 1, & \text{if } i = 1, \\ 0, & \text{otherwise,} \end{cases}$$

for each site $i \in [n]$ and route $j \in [m]$.

The first observation regarding the feasible solution of the OPCF-EBF is that the the demand constraint violation u_{jt} is 0, and that the number of depot BEBs η_j equals 1, for every route $j \in [m]$ and interval $t \in [0 : 1]$. Indeed, the objective function

$$Obj := \sum_{i=1}^{n} f_i x_i + \sum_{j=1}^{m} c_{beb} \cdot \eta_j + \sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij} z_{ij}^1 + \sum_{j=1}^{m} \sum_{t=0}^{1} c_u \cdot u_j^t$$

evaluated at the OPCF-EBF solution is such that $Obj \leq H + c_{beb} \cdot m$, by hypothesis, and from the choice of c_{beb} we have that $H < c_{beb}$. Thus, $Obj < c_{beb} \cdot (m + 1)$. Since c_u equals $c_{beb} \cdot (m + 1)$ this implies that u_{jt} is 0, for every route $j \in [m]$ and every time interval $t \in [0 : 1]$. The demand constraint $w_j^t + u_j^t \geq d_j^t$ at $t = 0$ implies that

$$\eta_j \geq w_j^0 \geq 1 - u_j^0 = 1, \qquad \forall j \in \mathcal{J}.$$

With this lower bound on η_j, we have that Obj satisfies $c_{beb} \cdot m \leq Obj < c_{beb} \cdot (m + 1)$, which implies that both η_j and w_j^0 must be equal to 1 for every route $j \in [m]$.

The second observation is that the number of working w_j^t and idling v_{js}^t depot BEBs satisfy

$$w_j^t = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{if } t = 1, \end{cases} \quad \text{and} \quad v_{js}^t = 0,$$

for every route $j \in [m]$, state of charge $s \in [1:2]$, and time interval $t \in [0:1]$. Indeed, because the depot BEBs have enough charge for only one time interval, all the buses must be charging at time $t = 1$ to be able to work again at time $t = 0$. This implies that $w_{j1} = 0$, for all $j \in [m]$. Consequently, the number of idling depot BEBs v_{js}^t is equal to 0, for all $j \in [m],\ s \in [1:2],\ t \in [0:1]$.

The third observation is that the solution (λ, π) defined by

$$\lambda_i = x_i, \quad \pi_{ij} = z_{ij}^1,$$

is feasible for the UFL problem with objective value K less than or equal to H. From the state transition dynamics $w_j^t + v_{j1}^t = \sum_{i=1}^n z_{ij}^{(t-1)} + v_{j1}^{(t-1)}$ at time $t = 1$, we conclude the identity $\sum_{i=1}^n z_{ij}^1 = 1$, for every route $j \in [m]$. It follows from $0 \leq y_i \leq m \cdot x_i$, $\beta_{ij}^t = z_{ij}^t$, and $\sum_{j \in [m]} \beta_{ij}^t \leq y_i$ the constraint $\sum_{j \in [m]} z_{ij}^1 \leq m \cdot x_i$, for every site $i \in I$. Note that z_{ij}^1 is a binary variable as there is only one BEB in each route. In particular,

$$K := \sum_{i=1}^n f_i \lambda_i + \sum_{i=1}^n \sum_{j=1}^m g_{ij} \pi_{ij} = Obj - c_{beb} \cdot m \leq H,$$

and this concludes the reduction proof.

We note that one can prove a similar reduction from the UFL to the single period *on-route BEB* only OPCF-EBF. □

2.6 Primal Heuristic Method

The OPCF-EBF model (2.1)-(2.9) turns out to be an extremely challenging large-scale integer linear program. The state-of-the-art commercial solver such as Gurobi cannot obtain a good feasible solution within a reasonable computation time as will be shown in the computation part. After explorations of various computation methods, it became evident the need for primal heuristics to warm-start Gurobi. In this section, we describe a primal heuristic called the *Policy Restriction*.

Policy Restriction heuristic.

This heuristic restricts the operation dynamics of depot BEBs to reduce the primal solution search. Indeed, we denote the set of positive demand time intervals as $T^{service}_{j,\theta} = \{t \in [0 : T-1] \mid d^{t,\theta}_j > 0\}$ and refer to it as the service times. Analogously, we define the set of zero-demand time intervals as $T^{off}_{j,\theta} = [0 : T-1] \backslash T^{service}_{j,\theta}$ and refer to it as off-service times.

The Policy Restriction heuristic prevents the depot BEBs from charging at any state s different from the depleted state $W_b + 1$ during the service times, that is,

$$z^{t,\theta}_{bijk} = 0, \qquad \forall t \in T^{service}_{j,\theta},\ s \in [2 : W_b],\ (b, i, j, k, \theta) \in \mathcal{B}_{depot} \times \mathcal{I} \times \mathcal{J} \times \mathcal{K} \times \Theta.$$

It also prevents the depot and on-route BEBs from being idle during service times, with the exception of depot BEBs when fully charged ($s = 1$):

$$v^{t,\theta}_{bjs} = 0, \qquad \forall t \in T^{service}_{j,\theta},\ s \in [2 : (W_b + 1)],\ (b, j, \theta) \in \mathcal{B}_{depot} \times \mathcal{J} \times \Theta,$$
$$\tilde{v}^{t,\theta}_{bjr} = 0, \qquad \forall t \in T^{service}_{j,\theta},\ r \in \mathcal{R}(j),\ (b, j, \theta) \in \mathcal{B}_{route} \times \mathcal{J} \times \Theta.$$

The idea of the above restriction is to use fully charged depot BEBs when it is most convenient in terms of cost. Lastly, the number of working depot and on-route BEBs must be zero during off-service times $t \in T^{off}_{j,\theta}$:

$$w^{t,\theta}_{bjs} = 0, \qquad \forall t \in T^{off}_{j,\theta},\ s \in [1:W_b],\ (b,j,\theta) \in \mathcal{B}_{depot} \times \mathcal{J} \times \Theta,$$

$$\widetilde{w}^{t,\theta}_{bjr} = 0, \qquad \forall t \in T^{off}_{j,\theta},\ r \in \mathcal{R}(j),\ (b,j,\theta) \in \mathcal{B}_{route} \times \mathcal{J} \times \Theta.$$

One advantage of the Policy Restriction (Policy-R) heuristic is that it always leads to a feasible solution.

Proposition 2.6.1. *The OPCF-EBF problem with the Policy-R constraints is feasible.*

Proof of Proposition 2.6.1. Consider a solution defined as follows:

- Define all the depot infrastructure x and y, depot BEBs η, and the associated operational variables w, v, and z as zero vectors.
- Define all the on-route infrastructure χ, on-route BEBs $\widetilde{\eta}$, and the associated operational variables $\widetilde{w}$ and $\widetilde{v}$ as zero vectors as well.
- Let the conventional buses ξ be such that it satisfies the retirement targets $\xi^{\theta}_{LB,j} \leq \xi^{\theta}_{j} \leq \xi^{\theta}_{UB,j}$ and the monotonicity constraints $\xi^{\theta}_{j} \leq \xi^{\theta-1}_{j}$ for all routes $j \in J$, and all investment periods $\theta \in \Theta$. Define the working conventional buses $\phi^{t,\theta}_{j}$ as zero and the idle conventional buses $\sigma^{t,\theta}_{j}$ as ξ^{θ}_{j} for all time intervals $t \in [0 : T-1]$, routes $j \in J$, and investment periods $\theta \in \Theta$.
- Let the demand slack variable $u^{t,\theta}_{j}$ be equal to $d^{t,\theta}_{j}$ for all time intervals $t \in [0 : T-1]$, routes $j \in J$, and investment periods $\theta \in \Theta$.

It is straightforward to check that this solution is feasible. Hence, the OPCF-EBF problem with Policy Restriction constraints is feasible. □

2.7 Case studies and analysis

Using our OPCF-EBF model, we present in Section 2.7.1 a bus electrification plan for the Metropolitan Atlanta Rapid Transit Authority (MARTA) of Atlanta and a battery sensitivity analysis in Section 2.7.2 for bus electrification of the Massachusetts Bay Transportation Authority (MBTA) of Boston using depot BEBs. In Section 2.7.3, we explain that the particular operation and fleet sizing for the Atlanta case study is also observed in the analytical solution of a simplified OPCF-EBF model with a single route, unlimited charging capacity, depot and on-route BEBs. We highlight in Section 2.7.4 the performance of our primal heuristic compared to Gurobi over 11 US and 2 non-US cities using real data. We also describe the cost and infrastructure involved in the transition to an entirely electric fleet with remarks in terms of fleet operation regarding the bus composition.

2.7.1 Atlanta MARTA case study: Bus electrification plan

The data used in this case study corresponds to a weekday bus schedule and it is based on the MARTA GTFS file available at [43] from August 2019, before the COVID-19 pandemic. In 2019, Atlanta had 110 bus routes, from which there were 115 terminal stops that could serve as possible locations to install on-route chargers. We assume an installation capacity of 2 on-route chargers per terminal station where each can serve up to 8 on-route BEBs each hour.

The bus garages operated by MARTA are taken as potential depot charging sites, with a total of five garages identified through [44], see the "D" marks in Figure 2.2. We use geospatial images to estimate the maximum installation capacity of depot chargers in each depot. The only depot charger considered in this study is a 70kW AC charger that costs $60.05k. The on-route 325kW DC charger costs $877.59k and both values comprise purchase, installation, and maintenance over 10 years [45].

We consider two models of BEBs in our studies. The first model is the New Flyer 40-foot

BEB with a 160 kWh battery capacity, 6 hours of operational capacity when fully charged, and it requires 3 hours to charge using the depot technology. The New Flyer BEB has the on-route charging capability and costs $943k each. The second model is the BYD 40-foot BEB with a 351 kWh battery capacity. We assume the BYD BEB has a 12-hour operational capacity and it requires 6 hours to fully charge. However, the BYD model does not have the on-route charging capability and it costs $1,093k per unit.

Atlanta bus fleet electrification plan.

The solution of our model gives an annual investment plan in depot and on-route chargers and BEBs over 10 years, summarized in Table 2.1. The investment plan is guided by the conventional bus retirement targets based on [46], which is column '# Conv. buses" (e.g. -5 means retiring 5 conventional buses). All other columns are from our numerical solution.

Table 2.1: Investment plan for MARTA on charging facilities and bus fleet units.

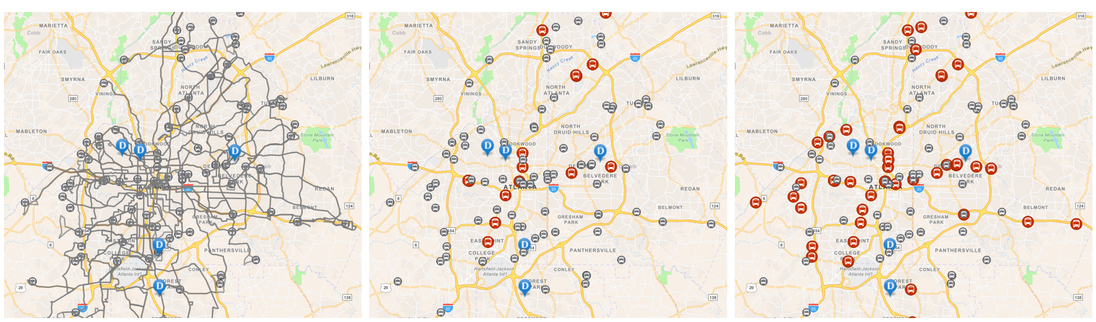
Year	# Depot BEBs	# On-route BEBs	# Conv. buses	# Depot chargers	# On-route chargers	Invest. cost ($ Million)	Op. cost ($ Million)
0	2	3	-5	1	2	$6.53	$38.14
1	11	0	-11	5	0	$10.26	$36.76
2	66	6	-72	33	2	$67.18	$36.87
3	46	27	-73	35	13	$72.92	$37.08
4	1	72	-72	2	14	$69.06	$35.71
5	1	27	-28	2	6	$25.94	$34.32
6	0	23	-23	0	3	$19.05	$32.94
7	1	46	-45	0	4	$35.98	$31.72
8	0	50	-49	0	5	$37.21	$30.43
9	19	40	-63	0	11	$45.25	$29.98
Total	147	294	-441	78	60	$389.39	$343.95

One interesting observation from Table 2.1 is that, during the first four years (years 0-

3), investment is primarily on depot BEBs and chargers, but from year 4 onwards, the investment shifts towards on-route BEBs and chargers. A similar investment pattern is also observed in other cities, see Section 2.7.4.

Also from Table 2.1, the replacement factor of the conventional bus fleet is 1, that is, the total number of retired conventional buses is equal to the total number of the added depot and on-route BEBs. The yearly investment cost of such an investment plan remains below \$70 million, except in year 3, with the total investment cost equal to \$390 million. The total operational cost over 10 years is comparable to the investment cost.

The spatial distribution of on-route chargers from our numerical solution is depicted in Figure 2.2. The D markers represent the bus depots, the smallest circles represent the potential on-route charging locations, while the larger ones are the installed on-route chargers. Generally, the model suggests the installation of on-route chargers from the area of the greatest confluence of bus routes in the downtown area towards the periphery of the city as shown in years 3 and 9 in Figures 2.2b and 2.2c. The illustrations of Figure 2.2 are generated using the ArcGIS tool, see [47].

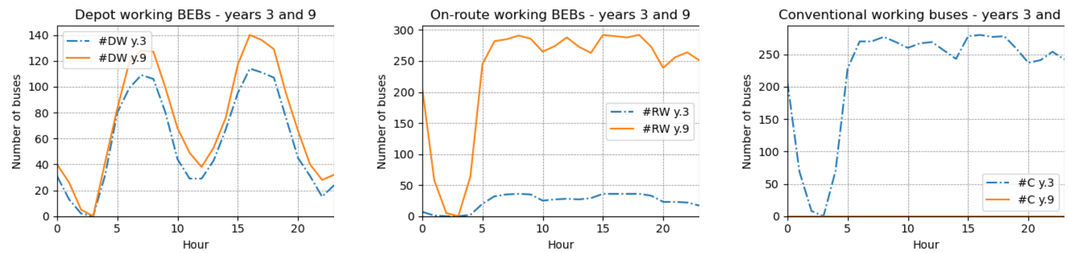


(a) Bus transportation network. (b) # Open stations: 16 – year 3. (c) # Open stations: 49 – year 9.

Figure 2.2: Spatial distribution of on-route charge stations for Atlanta.

Operation of the BEB fleet.

To understand how the mixed fleet of BEBs and conventional buses is operated by our model, we present the number of working depot BEBs, on-route BEBs, and conventional buses over 24 hours during investment years 3 and 9 in Figure 2.3. From Figure 2.3c, we note that the conventional buses meet part of the demand that is essentially constant throughout the day, named base demand, during investment year 3.



(a) Depot BEBs. (b) On-route BEBs. (c) Conventional buses.

Figure 2.3: Fleet operational dynamics over 24 hours per bus type.

Meanwhile, as seen in Figure 2.3a, the depot BEBs accommodate the rush hours fluctuation for both years 3 and 9. The most likely explanation is that the depot BEB New Flyer 40ft (160 KWh) is the cheapest option, its 6 hours battery performance is sufficient to cover each rush wave, and the 3 hours charging time is less than the in-between rush hour times. We observed that the number of BYD BEBs obtained in the solution is almost zero, which is possibly due to its purchase cost being slightly higher than the New Flyer (about 16% higher per BEB). The number of on-route BEBs from year 3 is not expressive in comparison with the total bus demand but in year 9 the on-route BEBs essentially replaced the conventional bus fleet from year 3, see Figures 2.3b and 2.3c. In summary, we observe that depot BEBs accommodate the variation in demand during rush hour waves, while on-route BEBs are responsible for handling the base demand.

2.7.2 Boston MBTA case study: Battery sensitivity analysis

In this section, we present a case study on the Massachusetts Bay Transportation Authority (MBTA) of Boston, Massachusetts. In the report [48], MBTA pointed out that their electrification strategy considers only depot BEBs and a type of diesel-electric hybrid bus. The justification for their strategy instead of an entirely electric fleet is that during the winter season the efficiency of a depot BEB drops to 4 hours of operation due to the use of heaters. MBTA's plan is to use hybrid buses to retire most of the old conventional diesel buses in order to meet the GHG reduction target set for 2030, [15].

Based on this scenario, we carry out a sensitivity analysis for the MBTA's 10-year investment plan, assuming *only* depot BEBs with battery performance values of 4, 6, 8, 10, and 12 hours, and a charging time of 4 hours. These numbers are based on the assumption that the insulation system and the battery capacity of electric buses may improve in the near future. The maximum demand for buses in this case study is 1108 buses and the result is summarized in Table 2.2. The column "Battery (h)" contains the battery performance in hours of operations for the depot BEBs; the column "# Depot BEBs" contains the number of depot BEBs needed to replace the conventional bus fleet, while maintaining the same level of service; the column "Ratio" is the ratio between the number of depot BEBs and the number of retired conventional buses; and the column "Opt. Gap" is the optimality gap of the solution found after 10 hours of computation. Note that the number of depot BEBs

Table 2.2: Sensitivity analysis of the operating capacity depot BEBs for the MBTA case study.

Battery (h)	# Depot BEBs	Ratio	Opt. Gap
4	1902	1.72	1.30%
6	1638	1.48	2.46%
8	1625	1.47	7.73%
10	1610	1.45	9.02%
12	1518	1.37	8.18%

needed to replace the conventional bus fleet decreases as the battery capacity increases.

To illustrate the need for extra depot BEBs, we present in Figure 2.4 a curve for the total number of working, charging, and idling depot BEBs with 8 hours of battery capacity. The curve of working depot BEBs is very close to the bus demand which indicates the same bus service level. But to compensate for the charging time, the depot BEBs require a coordinated operation that involves around 27% of the fleet constantly charging and the idle BEBs to start working at the specific times of day, as can be seen by the first and second rush waves.

Thus, one cannot expect a replacement ratio equal to 1 using exclusively depot BEBs if their

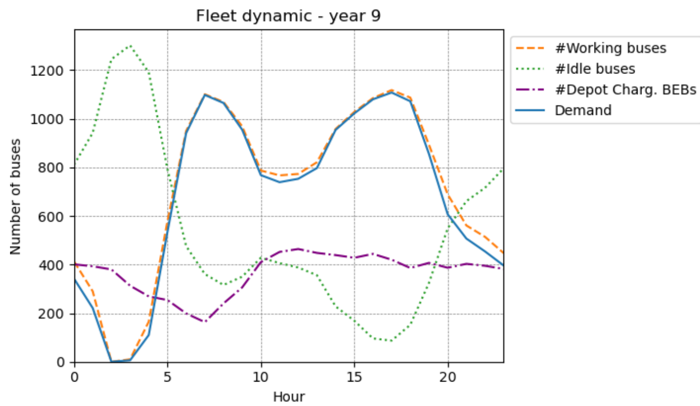


Figure 2.4: Depot BEBs' operational dynamics over 24 hours with a battery capacity of 8 hours.

battery capacity is not enough to operate through the entire service day. On the other hand, our mixed fleet solution involves a large proportion of on-route BEBs, but the deployment of such technology may delay the replacement of conventional buses and negatively impact the GHG reduction goal set for 2030, see [48]. Use of the diesel-electric hybrid buses is then a reasonable solution.

2.7.3 Analysis of the mixed depot and on-route fleet strategy

In this section, we provide an explanation for the BEB operation of our OPCF-EBF model through the analytical solution of a simplified model. Consider a simplified fleet sizing problem with only one route and one investment period, where the chargers and other infrastructure costs are aggregated into the BEB unit costs. We consider depot and on-route BEBs only, that is, no conventional buses in our fleet sizing problem. Suppose that in a *coarse* time discretization a depot BEB can only work for *one* time interval and need to fully charge on a consecutive interval.

Let $\eta, \widetilde{\eta}$ be the total number of the depot and on-route BEBs with unit costs c_d, c_r, respectively, and let w_t, v_t, and z_t be the number of working, idling, and charging depot BEBs at time t. Let $\widetilde{w}_t$ and $\widetilde{v}_t$ be the number of working and idling on-route BEBs, and let d_t be the bus demand at time t. Assume there are no charging or working costs. Then, our simplified fleet sizing model is

$$\begin{aligned}
\min \quad & c_d\eta + c_r\widetilde{\eta} & & & \\
\text{s.t.} \quad & \eta = w_0 + v_0 + z_0, & & \widetilde{\eta} = \widetilde{w}_0 + \widetilde{v}_0, & \\
& w_t + v_t = z_{t-1} + v_{t-1}, & & z_t = w_{t-1}, & \forall t \in [0 : T-1], \\
& \widetilde{w}_t + \widetilde{v}_t = \widetilde{w}_{t-1} + \widetilde{v}_{t-1}, & & w_t + \widetilde{w}_t \geq d_t, & \forall t \in [0 : T-1], \\
& \eta, \widetilde{\eta}, w_t, \widetilde{w}_t, v_t, \widetilde{v}_t, z_t \in \mathbb{Z}_+, & & & \forall t \in [0 : T-1].
\end{aligned} \tag{2.15}$$

The only important quantities to determine the optimal number of depot and on-route BEBs for (2.15) are $D_1 := \max_{t\in[0:T-1]} d_t$ and $D_2 := \max_{t\in[0:T-1]}(d_t + d_{t-1})$ as observed in the Proposition 2.7.1.

Proposition 2.7.1. *For every scenario of unit costs c_d and c_r, the optimal number of the depot and on-route BEBs to* (2.15) *and the corresponding numbers of working BEBs for each time interval $t \in [0 : T-1]$ is obtained in Table 2.3. The optimal solution of the variables z_t, v_t, and $\widetilde{v}_t$ is given by the relations $z_t = w_{t-1}$, $v_t = \eta - w_t - z_t$, and $\widetilde{v}_t = \widetilde{\eta} - \widetilde{w}_t$.*

Table 2.3: Optimal solution table of (2.15) for each objective coefficients c_r and c_d.

Coeff.	η	$\widetilde{\eta}$	w_t	$\widetilde{w}_t$
$c_r \leq c_d$	0	D_1	0	d_t
$c_r \geq 2c_d$	D_2	0	d_t	0
$c_d < c_r < 2c_d$	$2D_1 - D_2$	$D_2 - D_1$	$\max\{d_t + D_1 - D_2,\, 0\}$	$\min\{D_2 - D_1,\, d_t\}$

Proof of Proposition 2.7.1. We first simplify (2.15) by eliminating the charging variable z^t, and the idling variables v^t and $\widetilde{v}^t$. Indeed, we can replace z_t by w_{t-1} everywhere in (2.15) and this leads to the following model:

$$
\begin{aligned}
\min \quad & c_d\eta + c_r\widetilde{\eta} \\
\text{s.t.} \quad & \eta = w^0 + v^0 + w^{T-1}, \qquad \widetilde{\eta} = \widetilde{w}^0 + \widetilde{v}^0, \\
& w^t + v^t = w^{t-2} + v^{t-1}, && \forall t \in [0:T-1], \\
& \widetilde{w}^t + \widetilde{v}^t = \widetilde{w}^{t-1} + \widetilde{v}^{t-1}, \quad w^t + \widetilde{w}^t \geq d^t, && \forall t \in [0:T-1], \\
& \eta, \widetilde{\eta}, w^t, \widetilde{w}^t, v^t, \widetilde{v}^t \in \mathbb{Z}_+, && \forall t \in [0:T-1].
\end{aligned}
\tag{2.16}
$$

Note that $\eta = w^t + v^t + w^{t-1}$ is equivalent to $w^t + v^t = w^{t-2} + v^{t-1}$, for all $t \in [0:T-1]$, and that $\widetilde{\eta} = \widetilde{w}^t + \widetilde{v}^t$ is equivalent to $\widetilde{w}^t + \widetilde{v}^t = \widetilde{w}^{t-1} + \widetilde{v}^{t-1}$, for all $t \in [0:T-1]$. This leads to the following equivalent formulation:

$$
\begin{aligned}
\min \quad & c_d\eta + c_r\widetilde{\eta} \\
\text{s.t.} \quad & \eta = w^t + v^t + w^{t-1}, \qquad \widetilde{\eta} = \widetilde{w}^t + \widetilde{v}^t, && \forall t \in [0:T-1] \\
& w^t + \widetilde{w}^t \geq d^t, && \forall t \in [0:T-1], \\
& \eta, \widetilde{\eta}, w^t, \widetilde{w}^t, v^t, \widetilde{v}^t \in \mathbb{Z}_+, && \forall t \in [0:T-1].
\end{aligned}
\tag{2.17}
$$

From (2.17), it is straightforward to eliminate the idling variables v^t and $\widetilde{v}^t$. Let $v^t = \eta - w^t - w^{t-1}$ and $\widetilde{v}^t = \widetilde{\eta} - \widetilde{w}^t$, and because both variables are non-negative, we have the formulation below:

$$
\begin{aligned}
\min \quad & c_d\eta + c_r\widetilde{\eta} \\
\text{s.t.} \quad & \eta \geq w^t + w^{t-1}, \qquad \widetilde{\eta} \geq \widetilde{w}^t, && \forall t \in [0:T-1] \\
& w^t + \widetilde{w}^t \geq d^t, && \forall t \in [0:T-1], \\
& \eta, \widetilde{\eta}, w^t, \widetilde{w}^t \in \mathbb{Z}_+, && \forall t \in [0:T-1].
\end{aligned}
\tag{2.18}
$$

The lines of Table 2.3 induce feasible solutions to (2.18) with objectives c_rD_1, c_dD_2, and

$c_r(2D_1 - D_2) + c_d(D_2 - D_1)$. Consider the dual of the linear relaxation of (2.18):

$$\begin{aligned}\max\quad & \textstyle\sum_{t\in[0:T-1]} d^t\phi^t \\ \text{s.t.}\quad & \textstyle\sum_{t\in[0:T-1]} \pi_t \le c_d, && \textstyle\sum_{t\in[0:T-1]} \widetilde{\pi}^t \le c_r, \\ & -\pi^t - \pi^{t+1} + \phi^t \le 0, && -\widetilde{\pi}^t + \phi^t \le 0, && t \in [0:T-1], \\ & \pi^t, \widetilde{\pi}^t, \phi^t \ge 0, && && t \in [0:T-1],\end{aligned} \tag{2.19}$$

and let $a, b \in [0 : T-1]$ be such that $D_1 = d^a$ and $D_2 = d^b + d^{b-1}$. We use the Kronecker delta vectors δ^a and δ^b to define the dual feasible solutions, where

$$(\delta^a)^t := \begin{cases} 1, & \text{if } t = a, \\ 0, & \text{otherwise.} \end{cases}$$

One can check that the lines of Table 2.4 induce feasible solutions to the dual problem (2.19) with the same objective values c_rD_1, c_dD_2, and $c_r(2D_1 - D_2) + c_d(D_2 - D_1)$. Therefore, the solutions of Table 2.3 are optimal to (2.18).

Table 2.4: Optimal solutions of the dual problem (2.19) for each objective coefficients c_r and c_d.

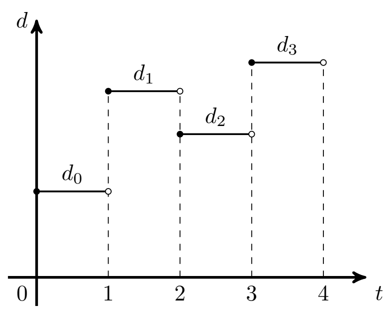
Coeff.	$\phi^t = \widetilde{\pi}^t$	π^t
$c_r \le c_d$	$c_r\delta^a$	$c_r\delta^a$
$c_r \ge 2c_d$	$c_d(\delta^b + \delta^{b-1})$	$c_d\delta^b$
$c_d < c_r < 2c_d$	$(2c_d - c_r)\delta^a + (c_r - c_d)(\delta^b + \delta^{b-1})$	$(2c_d - c_r)\delta^a + (c_r - c_d)\delta^b$

□

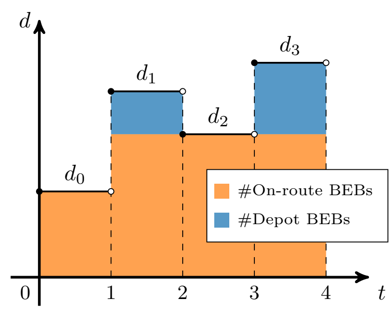
The following example provides the intuition behind the operational decision of our numerical experiments. Consider a 24-hour partition given by 4 time intervals: early morning $t = 0$, morning rush $t = 1$, inter-rush $t = 2$, and evening rush $t = 3$. On this timescale, it is reasonable to assume that a depot BEB can work during only one time interval and needs to fully charge in a consecutive interval. Suppose the demand $\{d_t\}_{t=0}^{3}$ is such that

$d_0 < d_2 < d_1 < d_3$ and $d_3 - d_2 < d_1 - d_0$, which is similar to the buses' rush waves, see Figure 2.5a. Then, $D_1 = d_3$ and $D_2 = d_2 + d_3$.

It is a reasonable approximation to assume that the deployment of an on-route BEB is more expensive than that of a depot BEB but less expensive than that of two depot BEBs, i.e., $c_d < c_r < 2c_d$, since the cost of an on-route charger can be divided equally among the on-route BEBs. This implies that the optimal fleet is $\eta = d_3 - d_2$ and $\widetilde{\eta} = d_2$ and the optimal working BEBs are given by $w_t = \max\{d_t - d_2, 0\}$ and $\widetilde{w}_t = \min\{d_2, d_t\}$, for each $t \in [0 : 3]$, see the illustration of Figure 2.5b. Thus, the optimal operation is to use on-route BEBs for the base demand and depot BEBs for the rush wave fluctuations. In reality, the OPCF-EBF solution may suggest more depot BEBs since there may exist many routes without common terminals, which increases the unit cost c_r.



(a) Simplified demand.



(b) Optimal working BEBs.

Figure 2.5: Bus demand and the optimal number of working BEBs if $c_d < c_r < 2c_d$.

2.7.4 Multi-city study and analysis

We benchmark the efficiency of the Policy Restriction heuristic with respect to the Gurobi solver's internal heuristic over a total execution time of four hours for 17 public transit systems with 11 US cities and 2 non-US cities, see Table 2.5.

Below are some comments on the results of the heuristics presented in Table 2.5.

Original: The Gurobi solver without warm-start have a gap greater than 90% for 6 of the 16 instances and an average gap of 49.9%, even after four hours of simulation.

Policy-R: The Policy Restriction heuristic has a much lower optimality gap with quite stable

Table 2.5: Primal heuristics optimality gap after 4 hours of computation.

City	Gurobi gap	Policy-R gap	# Depot BEBs	# On-route BEBs	# Depot charg-ers	# On-route charg-ers	Invest. cost ($ Mil-lion)
Chicago	99.71%	7.62%	874	598	380	96	1256.15
Dallas	100.0%	11.23%	251	348	125	52	519.07
Houston	70.55%	8.78%	580	316	156	50	753.65
Las Vegas	3.64%	1.82%	46	233	15	36	249.94
Los Angeles	99.21%	3.00%	1064	706	536	108	1507.30
NY (Bronx)	23.59%	8.23%	655	570	354	80	1046.20
NY (Brooklyn)	59.15%	8.42%	1481	1461	795	193	2512.65
NY (Manhatt.)	26.73%	4.32%	525	551	194	80	921.03
NY (Queens)	1.95%	6.08%	595	303	178	45	753.42
NY (St. Island)	0.32%	0.29%	565	52	105	6	498.63
San Francisco	3.95%	0.47%	745	572	169	82	1114.21
Seattle	3.61%	2.12%	169	26	29	4	159.01
Philadelphia	99.74%	7.56%	637	387	299	72	914.03
San Jose	7.15%	2.61%	209	198	48	42	406.33
Sydney	99.70%	3.84%	2249	726	838	115	2640.40
Toronto	99.75%	3.35%	982	808	456	154	1638.11
Washington DC	99.69%	3.24%	869	352	293	93	1058.05
Average Gap	52.85%	4.75%					
Std. Gap	43.33%	3.01%					

results overall. All instances of the Policy Restriction heuristic have a gap smaller than 10% and the average gap is only 4.48%.

Thus, the Policy Restriction heuristic proved to be the most reliable in terms of the optimality gap, which is why we used it in all our case studies. All the numerical experiments were performed on a cluster with 86 processors Intel Xeon Skylake and 317 Gb of shared RAM memory.

We also analyze some important stylized facts regarding the Policy-R primal solution. In Table 2.5 we have the total number of depot BEBs, on-route BEBs, depot chargers, on-route chargers, and total investment cost to preserve the same bus service level in those major cities. We assume for all instances a constant budget in every investment year and a conventional bus target of 0 at the last year.

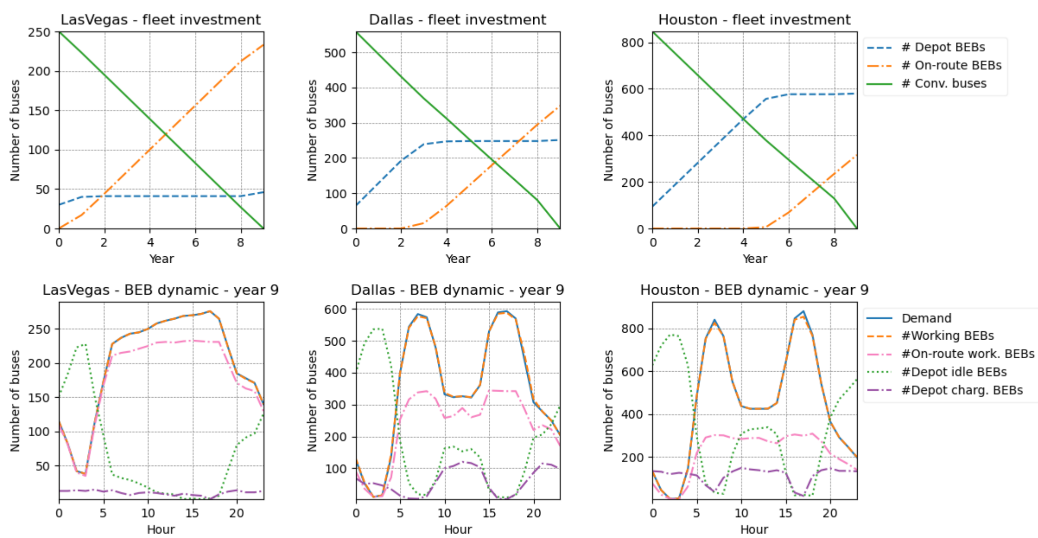


Figure 2.6: Multicity case study - fleet investment over 10 years.

The proportion of depot and on-route BEBs from Table 2.5 is primarily explained by the bus demand shape of each instance. Indeed, we show in Figure 2.6 the fleet investment evolution over 10 years and the BEB operation in year 9 for the cities of Dallas, Houston, and Las Vegas. We observe that the shape of the bus demand defines base demand which is fulfilled by on-route BEBs while the depot BEBs supply the difference to cover the bus service. As a secondary influence, we have the spatial distribution of the routes which may hinder the use of on-route chargers and increase the gap between on-route BEBs and the base demand, see Houston in Figure 2.6.

In the investment fleet evolution from Figure 2.6, we note the preference for depot BEBs in the early investment years until a saturation point and then the investment in on-route BEBs. This observation is consistent with the intuition that depot BEBs are cheaper to deploy than on-route BEBs and because of the constant discount factor $\gamma_\theta \in \{0, 1\}$ depot BEBs should be invested first. See Proposition 2.7.2 for a mathematical explanation of the role of the discount factor in ordering decisions.

Proposition 2.7.2. *Let* $c_1 < c_2 < \cdots < c_n$ *and* $0 < \beta < 1$ *be given. Let* $\pi : [n] \to [n]$ *denote a permutation, i.e. a bijection, where* $[n] := \{1, 2, \ldots, n\}$*. Then, the following minimization problem*

$$\min_{\substack{\pi:[n]\to[n] \\ \pi \text{ permutation}}} \sum_{\theta=1}^{n} \beta^{\theta-1} c_{\pi(\theta)}$$

has a unique optimal solution given by the identity permutation $\pi^*(\theta) = \theta$ *for all* $\theta \in [n]$*. That is, the minimum sum of a sequence of distinct numbers discounted by* β *is achieved by the increasing ordering of the numbers.*

Proof of Proposition 2.7.2. The solution to this problem can be find by induction. Indeed, the case $n = 1$ and $n = 2$ are trivial. Given a permutation π, we create another permutation $\widehat{\pi}$ by swapping two numbers:

$$\widehat{\pi}(i) = \begin{cases} n, & \text{if } i = n, \\ \pi(n), & \text{if } i = \pi^{-1}(n), \\ \pi(i), & \text{if } i \neq n, \pi^{-1}(n). \end{cases}$$

Note that the new permutation $\hat{\pi}$ is identical to the original permutation π except at two places: $\hat{\pi}(n) = n$, whereas $\pi(n) = i$, and $\hat{\pi}(i) = \pi(n)$, whereas $\pi(i) = n$.

Let $r = \pi^{-1}(n)$, and note that

$$\begin{aligned} & c_{\widehat{\pi}(r)} \cdot \beta^{r-1} + c_{\widehat{\pi}(n)} \cdot \beta^{n-1} < c_{\pi(r)} \cdot \beta^{r-1} + c_{\pi(n)} \cdot \beta^{n-1}, \\ \iff \quad & c_{\pi(n)}\beta^{r-1} + c_n \cdot \beta^{n-1} < c_n \cdot \beta^{r-1} + c_{\pi(n)} \cdot \beta^{n-1}, \\ \iff \quad & \beta^{n-1}(c_n - c_{\pi(n)}) < \beta^{r-1}(c_n - c_{\pi(n)}), \end{aligned}$$

where the last inequality holds since n is greater than r. Therefore,

$$\sum_{i=1}^{n-1} c_{\widehat{\pi}(i)}\beta^{i-1} + c_n\beta^{n-1} < \sum_{i=1}^{n} c_{\pi(i)}\beta^{i-1},$$

and because $\widehat{\pi}$ restricted to $[n-1]$ defines a permutation in $[n-1]$, we conclude the result by the induction hypothesis:

$$\sum_{i=1}^{n-1} c_i\beta^{i-1} + c_n\beta^{n-1} < \sum_{i=1}^{n} c_{\pi(i)}\beta^{i-1}.$$

□

2.8 Conclusions

In this paper, we proposed a novel investment planning model for the electrification of bus fleets and the building up of charging infrastructure for public transit systems. We present two real-world case studies and a multi-city analysis that demonstrate the effectiveness of our model. In the Atlanta case study, we presented an investment plan that achieves a 1:1 replacement ratio of conventional buses and sheds light on the operation of a bus fleet in transition. In the Boston case study, we assessed the sensitivity of the bus electrification plan with regard to BEB charging times, motivated by the significant weather-induced battery performance change in Boston winters. In the multi-city analysis, we observed that the proportion of depot and on-route BEBs is primarily dictated by the shape of the total bus demand curve. We proved that the OPCF-EBF model is NP-hard from a reduction of the

UFL problem. We developed the “Policy-Restriction” primal heuristic, which significantly outperformed Gurobi without warm-start in all of our instances. Overall, the proposed model, algorithms, and analysis provide a valuable tool to facilitate public transit authorities to carry out one of the most important and challenging tasks facing modern society, namely to electrify transportation in a timely and efficient manner.

CHAPTER 3

A POLYNOMIAL TIME SOLVABLE CLASS: THE FLEET SIZING PROBLEM

3.1 Introduction

According to Theorem 2.5.1, the growing numbers of bus routes and charging depots lead to computational intractability of OPCF-EBF, even with a single investment period and two battery states. In this chapter, we explore another dimension of the model with only one bus route and only depot BEBs, but an arbitrary number of battery states. We call this a fleet-sizing problem.

We show that, under a simple non-preemptive charging strategy with no early charging and idling, the fleet-sizing problem is polynomially solvable. The proof of this fact relies on the "almost" total unimodular nature of the constraint matrix for the operation problem given the number of depot BEBs. We also rely on a proximity result that quantifies the distance of the optimal solution to the linear relaxation solution $\eta_{*,LR}$. The polynomial-time complexity follows from the solution of a fixed number of linear programs.

Consider the following fleet sizing problem:

$$\min_{\eta} \quad \sum_{\theta \in \Theta} c_{\eta}^{\theta} \cdot \eta^{\theta} + \widetilde{F}_{\theta}(\eta^{\theta}) \tag{3.1a}$$

$$\text{s.t.} \quad \eta^{\theta-1} \leq \eta^{\theta}, \qquad \eta_{LB}^{\theta} \leq \eta^{\theta} \leq \eta_{UB}^{\theta}, \qquad \eta^{\theta} \in \mathbb{Z}_{+}, \qquad \theta \in \Theta. \tag{3.1b}$$

Here, $\widetilde{F}_{\theta}(\eta^{\theta})$ is the operational cost of a depot BEB fleet of size η^{θ} during the investment period θ, which is given by the following operational problem

$$\widetilde{F}_\theta(\eta^\theta) = \min_{w,z} \quad \sum_{t\in[0:T-1]} \left(\sum_{s=1}^{W} p_{w,s}^{t,\theta} \cdot w_s^{t,\theta} + p_z^{t,\theta} \cdot z^{t,\theta} \right) \tag{3.2a}$$

$$\text{s.t.} \quad \sum_{s=1}^{W} w_s^{0,\theta} + \sum_{l=0}^{L-1} z^{-l,\theta} = \eta^\theta, \tag{3.2b}$$

$$w_1^{t,\theta} = z^{t-L,\theta}, \quad w_s^{t,\theta} = w_{s-1}^{t-1,\theta}, \quad z^{t,\theta} = w_W^{t-1,\theta}, \;\; t \in [0:T-1], \tag{3.2c}$$

$$\sum_{s=1}^{W} w_s^{t,\theta} \geq d^{t,\theta}, \qquad t \in [0:T-1], \tag{3.2d}$$

$$w_s^{t,\theta},\ z^{t,\theta} \in \mathbb{Z}_+, \qquad t \in [0:T-1],\ s \in [1:W]. \tag{3.2e}$$

The objective in (3.2a) is to minimize the total working and charging costs, subject to the total number of depot BEBs equal to η^θ in (3.2b), the simple operation policy (3.2c) that requires a bus to work non-stop until it reaches the depleted battery state W (i.e. no early charging), that is, a bus must start charging and resume operation immediately after it is fully charged (i.e. non-preemptivie and no idling), and the working buses must meet the demand constraint (3.2d).

3.2 Contributions

1. **Tight LP relaxation of the operation problem:** The coefficient matrix of the operation problem may not be totally unimodular (TU). However, interestingly, by exploiting the rich symmetry imposed by the non-preemptive charging policy and the modular arithmetic, we can reformulate and unimodularly transform the operation problem to an equivalent formulation that does have the TU property.

2. **The fleet sizing problem as a Separable Convex Integer Program:**

   (a) Using the previous result, we extend the value function of the operation problem to an extended-real-valued convex piecewise linear function. Thus, the fleet sizing problem is essentially a separable convex integer program (SCIP), separable over the investment periods.

(b) Underlying this result is a proximity theorem proved for general SCIP that an optimal integer solution of SCIP belongs to the integer lattice of a ball centered at the LP relaxation's optimal solution. The key result shows that the search over the integer lattice can be further reformulated as a new integer linear program, which has an exact LP relaxation.

(c) Finally in the last step, we bound the number and size of all the LPs involved, and refer to the arithmetic complexity of an algorithm of Vaidya [12] to conclude the polynomial-time complexity of the fleet sizing problem.

3. **The Dyadic Contiguous Row (DCR) matrix:** We introduce a special class of separable integer program with totally unimodular constraints that can serve as a two stage decomposition method to prove polynomial solvability. In fact, we define the notion of a Dyadic Contiguous Row (DCR) matrix which extends the definition of a row-circular matrix of Bartholdi [6]. Given that the second stage integer program has a DCR coefficient matrix, the same complexity analysis performed for the fleet sizing problem applies.

The Chapter 3 is organized as follows. In Section 3.3, we describe the key ideas and results that motivated our polynomial-time algorithm for the fleet sizing problem (3.1). We prove in Section 3.3.1 that the operational problem value function $\widetilde{F}_\theta(\eta^\theta)$ can be extended to a tight piecewise linear convex function. In Section 3.3.2, we prove the correctness of a proximity-based reformulation which is the cornerstone of our polynomial-time algorithm. In Section 3.3.3, we bound the size of the intermediate linear programs and prove the complexity of our algorithm.

3.3 A polynomial time algorithm for the Fleet Sizing Problem

We outline the results that prove the polynomial solvability of the fleet sizing problem (3.1). The building blocks of our polynomial-time algorithm are the Lemma 3.3.1 that provides key properties of the value function $\widetilde{F}_\theta$ and the proximity-based reformulation (3.6) which is a type of binarization based on the closedness of optimal integral and linear relaxation solutions for a class of Separable Integer Convex Programs. We state our polynomial-time algorithm, Algorithm 1, and summarize the computational complexity necessary to solve all the intermediate linear programs and find the optimal integral solution to (3.1).

Lemma 3.3.1. *The domain of $\widetilde{F}_\theta$ is contained in the set of multiples of $(W+L)/k$, where k is the greatest common divisor of $W+L$ and T:*

$$\mathrm{dom}(\widetilde{F}_\theta) \subseteq \left\{ \frac{i(W+L)}{k} \in \mathbb{Z} \;\middle|\; i \in \mathbb{Z} \right\}. \tag{3.3}$$

In particular, the linear relaxation of (3.2) *is a tight convex lower approximation of $\widetilde{F}_\theta(\eta^\theta)$.*

Lemma 3.3.1 motivates the change of variables $\eta^\theta = \frac{W+L}{k} \cdot \overline{\eta}^\theta$. Let the new objective cost be $c_{\overline{\eta}}^\theta = \frac{W+L}{k} c_\eta^\theta$, let the new lower and upper bounds be $\overline{\eta}_{LB}^\theta = \left\lceil \eta_{LB}^\theta \cdot \frac{k}{W+L} \right\rceil$ and $\overline{\eta}_{UB}^\theta = \left\lfloor \eta_{UB}^\theta \cdot \frac{k}{W+L} \right\rfloor$, and let the new value function be $\overline{F}_\theta(\overline{\eta}^\theta) = \widetilde{F}_\theta\left(\frac{W+L}{k} \cdot \overline{\eta}^\theta\right)$. Thus, the original fleet sizing problem (3.1) can be reformulated as

$$\min_{\eta} \quad \sum_{\theta \in \Theta} c_{\overline{\eta}}^\theta \cdot \overline{\eta}^\theta + \overline{F}_\theta(\overline{\eta}^\theta) \tag{3.4a}$$

$$\text{s.t.} \quad \overline{\eta}^{\theta-1} \le \overline{\eta}^\theta, \qquad \overline{\eta}_{LB}^\theta \le \overline{\eta}^\theta \le \overline{\eta}_{UB}^\theta, \qquad \overline{\eta}^\theta \in \mathbb{Z}_+, \qquad \theta \in \Theta. \tag{3.4b}$$

The main difference between (3.1) and (3.4) is that the domain of $\overline{F}_\theta$ is a subset set of the integers instead of the multiples of $(W+L)/k$.

Also from Lemma 3.3.1, we extend the value function $\widetilde{F}_\theta$ to a convex function using the linear relaxation of (3.2). So, without loss of generality, we assume that $\widetilde{F}_\theta$ is a polyhedral function (proper piecewise linear convex function). This implies that (3.4) is a *Separable Convex Integer Program* over totally unimodular constraints. This additional structure allows us to use a Proximity Theorem to significantly restrict the search for an optimal integral solution.

Indeed, we obtain the optimal solution to (3.4) by formulating an auxiliary problem that works as a local search. Let $\overline{\eta}_{*,LR}$ be an optimal solution to the linear relaxation of (3.4), where $\overline{F}_\theta(\cdot)$ is extended to fractional values of $\overline{\eta}^\theta$ by considering the linear relaxation of (3.2). Let h^θ_{LB} and h^θ_{UB} be the minimum and maximum integer $h \in [0 : 2|\Theta|]$ such that $\overline{F}_\theta(\lfloor\overline{\eta}^\theta_{*,LR}\rfloor - |\Theta| + h) < \infty$, respectively, and let $q^\theta_{\delta,h}$ be the cost vector defined as

$$q^\theta_{\delta,h} = \begin{cases} \overline{F}_\theta(\lfloor\overline{\eta}^\theta_{*,LR}\rfloor - |\Theta| + h^\theta_{LB}), & \text{if } h = h^\theta_{LB}, \\ \overline{F}_\theta(\lfloor\overline{\eta}^\theta_{*,LR}\rfloor - |\Theta| + h) - \overline{F}_\theta(\lfloor\overline{\eta}^\theta_{*,LR}\rfloor - |\Theta| + h - 1), & \text{if } h \in [h^\theta_{LB} + 1 : h^\theta_{UB}], \\ 0, & \text{if } h \notin [h^\theta_{LB} : h^\theta_{UB}], \end{cases} \tag{3.5}$$

for all $h = 0, \ldots, 2|\Theta|$, and $\theta \in \Theta$. Theorem 3.3.1 guarantees that a binarization of the infinity norm ball of radius $2|\Theta|$ centered at $\overline{\eta}_{*,LR}$ contains the optimal solution for (3.4) and the corresponding reformulation also has a total unimodular coefficient matrix.

Theorem 3.3.1. *The separable convex integer program* (3.1) *can be reformulated as the* *following integer linear program:*

$$\min_{\overline{\eta},\delta} \sum_{\theta\in\Theta}\left(c_{\overline{\eta}}^{\theta}\cdot\overline{\eta}^{\theta}+\sum_{h=0}^{2|\Theta|}q_{\delta,h}^{\theta}\cdot\delta_h^{\theta}\right) \tag{3.6a}$$

$$\text{s.t. } \overline{\eta}^{\theta-1}\le\overline{\eta}^{\theta},\qquad \overline{\eta}_{LB}^{\theta}\le\overline{\eta}^{\theta}\le\overline{\eta}_{UB}^{\theta},\qquad \theta\in\Theta, \tag{3.6b}$$

$$\overline{\eta}^{\theta}-\sum_{h=0}^{2|\Theta|}\delta_h^{\theta}=\lfloor\overline{\eta}_{*,LR}^{\theta}\rfloor-|\Theta|-1,\qquad \theta\in\Theta, \tag{3.6c}$$

$$\delta_h^{\theta}=1,\qquad h\in[0:h_{LB}^{\theta}],\ \theta\in\Theta, \tag{3.6d}$$

$$\delta_h^{\theta}=0,\qquad h\in[h_{UB}^{\theta}+1:2|\Theta|],\ \theta\in\Theta, \tag{3.6e}$$

$$\overline{\eta}^{\theta}\in\mathbb{Z}_+,\ \delta_h^{\theta}\in\{0,1\},\qquad h=0,\dots,2|\Theta|,\ \theta\in\Theta. \tag{3.6f}$$

In particular, an optimal solution $(\overline{\eta}_*,\delta_*)$ *to* (3.6) *exists if and only if an optimal solution to* (3.1) *exists. The constraint matrix induced by* (3.6b)-(3.6f) *is totally unimodular.*

It follows from Theorem 3.3.1 that the fleet sizing problem (3.6) is polynomially solvable since the constraint matrix is totally unimodular. See Algorithm 1 for the complete description of the Proximity algorithm.

Algorithm 1 Proximity Algorithm

Find an optimal basic feasible solution $(\overline{\eta}_{*,LR},\overline{\zeta}_{*,LR})$ to the linear relaxation of (3.4).
for $\theta\in\Theta$ **do**
    for $h=0,1,\dots,2|\Theta|$ **do**
        Let $\overline{\eta}^{\theta}=\lfloor\overline{\eta}_{*,LR}^{\theta}\rfloor-|\Theta|-1+h$
        Compute the optimal value $\overline{F}_{\theta}(\overline{\eta}^{\theta})=\widetilde{F}_{\theta}\left(\frac{W+L}{k}\cdot\overline{\eta}^{\theta}\right)$ of problem (3.2).
        Define $q_{\delta,h}^{\theta}$ according to the expression (3.5).
Solve the proximity problem (3.6) using $q_{\delta,h}^{\theta}$ and $\lfloor\overline{\eta}_{*,LR}^{\theta}\rfloor$ as inputs.
return an optimal fleet size $\eta_*=\frac{W+L}{k}\cdot\overline{\eta}_*$.

The time complexity of the Proximity algorithm is $O\big((k^3|\Theta|^3+|\Theta|^6)\mathbb{L}\big)$, as described in Theorem 3.3.2, where $\mathbb{L}$ is the size of the integer programming instance (3.1) as defined in Section 3.3.3, Equation (3.37).

Theorem 3.3.2. *The time complexity to obtain an optimal integral solution* η^* *to* (3.1) *using the Algorithm 1 is* $O\big((k^3|\Theta|^3 + |\Theta|^6)\mathbb{L}\big)$ *arithmetic operations at a precision of* $O(\mathbb{L})$ *bits.*

3.3.1 Properties of the Operational Problem with Simple Charging Policy

We detail in this section the intermediate results that lead to the proof of Lemma 3.3.1. The first is a variable reduction reformulation implied by the battery state transition and recharging equations (3.2c). In this reformulation, we introduce a new variable with a different equivalence class index which is induced by the greatest common divisor of $(W + L)$ and T. The new index refers to the possible alignments between BEB operation plus charging cycles and the periodic planning horizon. Then, we provide a symmetry breaking unimodular transformation which reveals the almost TU property of the constraint matrix in our second reformulation. Since all the transformations are one-to-one and preserve integrality we conclude Lemma 3.3.1.

Below, we have our variable reduction reformulation for the operational problem (3.2).

Lemma 3.3.2 (Variable Reduction)**.** *Let* k *be the greatest common divisor of* $(W + L)$ *and* T*. The problem* (3.2) *is equivalent to the following model in* ζ *variables only*

$$\widetilde{F}(\eta) = \min_{\zeta} \quad \sum_{i=0}^{k-1} p_{\zeta,i} \cdot \zeta_i \tag{3.7a}$$

$$\text{s.t.} \quad \sum_{i=0}^{k-1} \zeta_i = \frac{k}{W+L} \cdot \eta, \tag{3.7b}$$

$$\sum_{l=0}^{W-1} \zeta_{i-l} \geq \widetilde{d}_i, \qquad i \in [0 : k-1], \tag{3.7c}$$

$$\zeta_i \in \mathbb{Z}_+, \qquad i \in [0 : k-1], \tag{3.7d}$$

where the ζ *indexes* i*'s are equivalence classes modulo* k*. The coefficients* $p_{\zeta,i}$ *and* $\widetilde{d}_i$ *are* *defined as*

$$p_{\zeta,i} = \sum_{\substack{t\in[0:T-1]\\ t\%k=i}} \left[p_z^t + \sum_{\substack{\tau\in[0:T-1],\ s\in[1:W],\\ \text{s.t. } \tau-s-L+1=t.}} p_{w,s}^{\tau} \right], \quad \textit{and} \quad \widetilde{d}_i = \max_{\substack{t\in[0:T-1]\\ t\%k=i}} d^{t+L}, \tag{3.8}$$

for all $i \in [0:k-1]$*, and* $t\%k$ *represents the remainder of* t *divided by* k*. The map between the feasible solutions from problem* (3.7) *and the original operational problem* (3.2) *is given by the relation*

$$z^t = \zeta_{t\%k}, \quad \textit{and} \quad w_s^t = z^{t-s-L+1}, \tag{3.9}$$

for all $s \in [1:W]$ *and* $t \in [0:T-1]$*. In particular, the fleet size* η *must be a multiple of* $(W+L)/k$*, otherwise the original operational problem* (3.2) *is infeasible.*

Proof of Lemma 3.3.2. First, it follows from (3.2c) that $w_s^t = z^{t-s-L+1}$ for all charge states $s \in [1:W]$ and time intervals $t \in [0:T-1]$. This reduces the original operational problem (3.2) to the following:

$$\widetilde{F}(\eta) = \min_z \quad \sum_{t\in[0:T-1]} \widetilde{p}_z^t \cdot z^t \tag{3.10a}$$

$$\text{s.t.} \quad \sum_{l=0}^{L+W-1} z^{-l} = \eta \tag{3.10b}$$

$$z^t = z^{t-W-L}, \qquad t \in [0:T-1], \tag{3.10c}$$

$$\sum_{l=0}^{W-1} z^{t-l-L} \geq d^t, \qquad t \in [0:T-1], \tag{3.10d}$$

$$z_t \in \mathbb{Z}_+, \qquad t \in [0:T-1], \tag{3.10e}$$

where the cost vector $\widetilde{p}_z$ is defined as

$$\widetilde{p}_z^t = p_z^t + \sum_{\substack{\tau\in[0:T-1],\ s\in[1:W],\\ \text{s.t. } \tau-s-L+1=t.}} p_{w,s}^{\tau}, \tag{3.11}$$

for all $t \in [0:T-1]$.

The equality constraint (3.10c) creates a symmetry, i.e. a periodicity of $W + L$, on the z-variable space. Moreover, recall that t is an equivalence class modulo T, so $t + yT$ is equal to t for all $y \in \mathbb{Z}$. This fact together with the constraint (3.10c) implies the equality

$$z^t = z^{t+x\cdot(W+L)+y\cdot T}, \tag{3.12}$$

for every $x, y \in \mathbb{Z}$, and every $t \in [0 : T-1]$. By the Bezout's identity, there are integers $\overline{x}$ and $\overline{y}$ such that $k = \overline{x}\cdot(W+L)+y\cdot T$, where k is the greatest common divisor of $(W+L)$ and T, and k is also the smallest positive integer given by any integral combination of $(W+L)$ and T. Thus, the number of distinct z variables is k, and the constraint (3.12) can be equivalently represented as $z^t = z^{t+a\cdot k}$, for every $a \in \mathbb{Z}$ and every $t \in [0 : T-1]$. Let ζ_i be defined as $\zeta_i = z^i$ for all $i \in [0 : k-1]$. Because of the identity $z^t = z^{t+a\cdot k}$, we have that

$$z^t = \zeta_{t\%k}, \tag{3.13}$$

so the total fleet constraint (3.10b) can be described in terms of ζ as

$$\eta = \sum_{l=0}^{W+L-1} z^{-l} = \sum_{l=0}^{\frac{(W+L)}{k}\cdot k-1} \zeta_{(-l)\%k} = \frac{(W+L)}{k}\cdot\sum_{i=0}^{k-1}\zeta_l. \tag{3.14}$$

The demand constraint (3.10d) in terms of the variables ζ becomes $\sum_{l=L}^{W+L-1}\zeta_{(t-l)\%k} \geq d_t$ for all $t \in [0 : T-1]$. So, by the change of variable $t := t + L$, and by taking a maximum of the right-hand side demand d^{t+L} over $t \in [0 : T-1]$ modulo k, i.e., $t\%k = i$, we obtain the following expression:

$$\sum_{l=0}^{W-1}\zeta_{i-l} \geq \max_{\substack{t\in[0:T-1] \\ t\%k=i}} d^{t+L}, \tag{3.15}$$

for all $t \in [0 : T-1]$. Finally, the cost $p_{\zeta,i}$ follows from (3.11) similarly by adding $\widetilde{p}_z^t$ over $t \in [0 : T-1]$ modulo K that is, $p_{\zeta,i} = \sum_{\substack{t\in[0:T-1] \\ t\%k=i}} \widetilde{p}_z^t$, for all $i \in [0 : k-1]$. $\square$

Although the reduced problem (3.7a) has a simpler structure compared to the original operational model (3.2), the new demand constraint (3.7c) is inconvenient to analyze. Indeed, the wraparound property of the indexes i's leads to a complicated expression for the summation $\sum_{l=0}^{W-1} \zeta_{i-l}$ in terms of $\zeta_0, \zeta_1, \dots, \zeta_{k-1}$ with coefficients that may be greater than 1. In order to improve the analysis we perform a symmetry-breaking transformation, this time with a unimodular linear transformation $R : \mathbb{R}^k \to \mathbb{R}^k$ defined as

$$(R\zeta)_i = \sum_{l=0}^{i} \zeta_l, \quad \forall 0 \le i \le k-1. \tag{3.16}$$

Note that its inverse R^{-1} is given by the formula

$$(R^{-1}\overline{\zeta})_i = \begin{cases} \overline{\zeta}_0, & \text{if } i = 0, \\ \overline{\zeta}_i - \overline{\zeta}_{i-1}, & \text{if } 1 \le i \le k-1. \end{cases} \tag{3.17}$$

Recall that a matrix is called *unimodular* if it is a square integral matrix with determinant $+1$ or -1. The unimodularity property holds for R since it is an integral lower triangular matrix with ones in its main diagonal.

Lemma 3.3.3 (Unimodular transformation)**.** *The change of variables $\overline{\zeta} := R\zeta$ applied to the reduced operational problem* (3.7) *results in the following problem:*

$$\widetilde{F}(\eta) = \min_{\overline{\zeta}} \quad \sum_{i=0}^{k-1} \overline{p}_{\zeta,i} \cdot \overline{\zeta}_i \tag{3.18a}$$

$$\textit{s.t.} \quad \overline{\zeta}_{k-1} = \frac{k}{W+L} \cdot \eta, \tag{3.18b}$$

$$\overline{\zeta}_i - \overline{\zeta}_{i-W} + \left(\left\lfloor \frac{W}{k} \right\rfloor + \mathbb{I}_{[i+1,\infty)}(W\%k) \right) \overline{\zeta}_{k-1} \ge \widetilde{d}_i, \quad i \in [0 : k-1], \tag{3.18c}$$

$$\overline{\zeta}_0 \ge 0, \tag{3.18d}$$

$$\overline{\zeta}_i - \overline{\zeta}_{i-1} \ge 0, \quad i \in [1 : k-1], \tag{3.18e}$$

$$\overline{\zeta}_i \in \mathbb{Z}, \quad i \in [0 : k-1], \tag{3.18f}$$

where $\mathbb{I}_{[i+1,\infty)}(x)$ is the indicator function that is 1 if x is greater than or equal to $i+1$, and 0 otherwise, and the cost coefficient $\overline{p}_{\zeta,i}$ is defined as

$$\overline{p}_{\zeta,i} = \begin{cases} p_{\zeta,i} - p_{\zeta,i+1}, & \textit{if } 0 \leq i \leq k-2, \\ p_{\zeta,k-1}, & \textit{if } i = k-1, \end{cases} \tag{3.19}$$

for all $i \in [0 : k-1]$. In particular, the polyhedron defined by the linear relaxation of (3.18) *is integral whenever η is a multiple of $(W+L)/k$. So, the linear relaxation value function $\widetilde{F}(\eta)$ is an extended real-valued convex piecewise linear function for continuous values of η.*

Proof of Lemma 3.3.3. Because $\zeta_i = \overline{\zeta}_i - \overline{\zeta}_{i-1}$, for every $i \in [1 : k-1]$, and $\zeta_0 = \overline{\zeta}_0$. we can describe the left-hand side of the constraint (3.7b) as $\sum_{i=0}^{k-1} \zeta_i = \overline{\zeta}_0 + \sum_{i=1}^{k-1}(\overline{\zeta}_i - \overline{\zeta}_{i-1}) = \overline{\zeta}_{k-1}$. Similarly for the left-hand side of (3.7c). Indeed,

$$\begin{aligned} \sum_{l=0}^{W-1} \zeta_{i-l} &= \sum_{l=0}^{k\left\lfloor \frac{W}{k} \right\rfloor - 1} \zeta_{i-l} + \sum_{l=k\left\lfloor \frac{W}{k} \right\rfloor}^{W-1} \zeta_{i-l} && \text{(3.20a)} \\ &= \left\lfloor \frac{W}{k} \right\rfloor \left(\zeta_i + \zeta_{i-1} + \cdots + \zeta_0 + \zeta_{k-1} + \zeta_{k-2} + \cdots + \zeta_{i+1}\right) && \text{(3.20b)} \\ &\quad + \sum_{l=k\left\lfloor \frac{W}{k} \right\rfloor}^{W-1} \zeta_{i-l} \\ &= \left\lfloor \frac{W}{k} \right\rfloor \overline{\zeta}_{k-1} + \sum_{l=k\left\lfloor \frac{W}{k} \right\rfloor}^{W-1} \zeta_{i-l}. && \text{(3.20c)} \end{aligned}$$

Since any integer W can be described as $W = k\left\lfloor \frac{W}{k} \right\rfloor + W\%k$, we have the following equalities for $\sum_{l=k\left\lfloor \frac{W}{k} \right\rfloor}^{W-1} \zeta_{i-l}$:

$$\sum_{l=k\left\lfloor \frac{W}{k} \right\rfloor}^{W-1} \zeta_{i-l} = \sum_{l=0}^{W\%k-1} \zeta_{i-l} = \begin{cases} \overline{\zeta}_i - \overline{\zeta}_{i-W\%k}, & \text{if } W\%k \leq i, \\ \overline{\zeta}_i - \overline{\zeta}_{i-W\%k} + \overline{\zeta}_{k-1}, & \text{if } W\%k \geq i+1, \end{cases} \tag{3.21}$$

where the last equality follows from noting that the term $\overline{\zeta}_{k-1}$ is added to the final expression whenever ζ_0 appears in a consecutive summation. The expression (3.18c) follows from (3.21) because of the identity $i - W\%k = (i-W)\%k$ and that we can drop the remainder operator $\%$ since the indexes of ζ and $\overline{\zeta}$ are equivalence classes modulo k. The expression (3.19) for the objective cost is straightforward.

Finally, we prove that the linear relaxation polyhedron induced by (3.18b)-(3.18f) is integral whenever η is a multiple of $(W+L)/k$. Indeed, the last variable $\overline{\zeta}_{k-1}$ is fixed and equal to $\frac{k}{W+L} \cdot \eta$, so we can replace it in every occurrence of $\overline{\zeta}_{k-1}$, which leads to an integral right-hand side vector. We conclude the integrality of the linear relaxation polyhedron by noting that the constraint matrix associated to the variables $\overline{\zeta}_0, \ldots, \overline{\zeta}_{k-2}$ is totally unimodular since it has at most one $+1$ and -1 at each row. □

We can now prove Lemma 3.3.1 using the properties of the reformulations.

Lemma 3.3.1. *The domain of $\widetilde{F}_\theta$ is contained in the set of multiples of $(W+L)/k$, where k is the greatest common divisor of $W+L$ and T:*

$$\operatorname{dom}(\widetilde{F}_\theta) \subseteq \left\{ \frac{i(W+L)}{k} \in \mathbb{Z} \,\middle|\, i \in \mathbb{Z} \right\}. \tag{3.3}$$

In particular, the linear relaxation of (3.2) *is a tight convex lower approximation of $\widetilde{F}_\theta(\eta^\theta)$.*

Proof of Lemma 3.3.1. From Lemmas 3.3.2 and 3.3.3, we know that the feasible solutions of the original operational problem (3.2) have a one-to-one correspondence with the feasible solutions of the reformulated model (3.18). So, it is straightforward to note that the original operational problem is infeasible when η^θ is not a multiple of $(W+L)/k$, which proves the inclusion (3.3).

Denote the polyhedron defined by the linear relaxation of the constraints (3.2b)-(3.2e) as $P_\theta(\eta^\theta)$. Then $P_\theta(\eta^\theta)$ is integral if and only if the minimum of

$$\min_{(w,z) \in P_\theta(\eta^\theta)} p_w^\top w + p_z^\top z \tag{3.22}$$

is either integral or $-\infty$, for every $p_w \in \mathbb{Z}^{W\times T}$ and $p_z \in \mathbb{Z}^T$. Using the variable reduction map from Lemma 3.3.2 and the unimodular change of variables from Lemma 3.3.3, we have that (3.22) can be the reduced to the following problem:

$$\min_{\overline{\zeta}} \quad \sum_{i=0}^{k-1} \overline{p}_{\zeta,i} \cdot \overline{\zeta}_i \tag{3.23a}$$

$$\text{s.t.} \quad (3.18b) - (3.18e) \tag{3.23b}$$

$$\overline{\zeta}_i \geq 0, \qquad i \in [0 : k-1]. \tag{3.23c}$$

Since $\overline{p}_{\zeta,i}$ is integral whenever $p^t_{w,s}$ and p^t_z are integral, and the constraints (3.18b)-(3.18e) induce an integral polyhedron by Lemma 3.3.3, we conclude that the optimal value of (3.22) is integral or $-\infty$. □

3.3.2 The Proximity Reformulation for Separable Convex Integer Programs

In this section, we address the proximity based reformulation stated in Theorem 3.3.1. In fact, such auxiliary problem is inspired by a proximity result for the class of Separable Convex Integer Programs with Totally Unimodular constraints which guarantees that an infinity-norm ball of radius n and centered in any linear relaxation solution contains an integral optimal solution, where n is the variable space dimension. The proximity-based reformulation (3.4) is a binarization of the variables space to represent the integral lattice inside the infinity-norm ball. Also, the reformulation (3.4) is an integral linear program. This implies that if the evaluation of the separable objective function takes a polynomial number of arithmetic operations and the corresponding feasible region is a polytope then Theorem 3.3.1 implies that the Separable Convex Integer Program with Totally Unimodular constraints is polynomially solvable.

Let $\{f_i\}_{i=1}^n$ be univariate real-valued convex functions, and consider the following separable convex integer programming problem:

$$\min \sum_{i=1}^{n} f_i(y_i) \tag{3.24a}$$

$$\text{s.t. } Ay \geq b, \tag{3.24b}$$

$$y \in \mathbb{Z}^n, \tag{3.24c}$$

where A is a totally unimodular (TU) matrix, and b is an integer vector. The goal of this section is to prove that we can use the linear relaxation to perform an efficient local search for an optimal integral solution. We assume the minimum of the linear relaxation of the integer program (3.24) exists and it is attainable. The feasibility of the integer program (3.24) is implied by the feasibility of the corresponding linear relaxation and the fact that A is TU and b is integral.

Even extended real-valued functions fit the scope of the program (3.24). Let f_i be an *extended* real-valued proper convex function of the form:

$$f_i(x) = \begin{cases} g_i(x), & \text{if } x \in [a_i, b_i], \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.25}$$

where $g_i(x)$ is a univariate real-valued convex function. A relevant example of such a function is the polyhedral function $\overline{F}_\theta(\cdot)$. Indeed, if $\{f_i\}_{i=1}^n$ are extended real-valued convex functions such as (3.25) we can reformulate (3.24) as follows:

$$\min \sum_{i=1}^{n} g_i(y_i) \tag{3.26a}$$

$$\text{s.t. } Ay \geq b, \tag{3.26b}$$

$$\lceil a \rceil \leq y \leq \lfloor b \rfloor, \tag{3.26c}$$

$$y \in \mathbb{Z}^n. \tag{3.26d}$$

Note that the constraint matrix induced by (3.26b)-(3.26c) is still TU, and the right-hand side vectors are still integral.

Theorem 3.3.3 (Proximity Theorem for Separable Convex Integer Programs)**.** *Suppose* $\{f_i\}_{i=1}^n$ *are convex proper real-valued functions and let* y^* *and* w^* *be optimal integral and continuous linear relaxation (LR) solutions to* (3.24)*, respectively. Then,*

1. *there exists an optimal* integral *solution* $\widehat{y}$ *to* (3.24) *such that* $\|\widehat{y} - w^*\|_\infty \leq n$*.*

2. *there exists an optimal LR solution* $\widehat{w}$ *to* (3.24) *such that* $\|y^* - \widehat{w}\|_\infty \leq n$*.*

Proof to Theorem 3.3.3. A slightly more general statement along with its proof can be found in [5]. □

The next result provides a method to solve separable convex integer programs assuming that the summation terms $\{f_i\}_{i=1}^n$ are cheap to evaluate. Let $h_{i,LB}$ and $h_{i,UB}$ be the minimum and maximum index $h \in \{0, 1 \ldots, 2n\}$ such that $f_i(\lfloor w_i^* \rfloor - n + h) < +\infty$, respectively, and let $q_{i,h}$ be the following objective cost:

$$q_{i,h} = \begin{cases} f_i(\lfloor w_i^* \rfloor - n + h_{i,LB}), & \text{if } h = h_{i,LB}, \\ f_i(\lfloor w_i^* \rfloor - n + h) - f_i(\lfloor w_i^* \rfloor - n + h - 1), & \text{if } h \in [h_{i,LB} + 1, h_{i,UB}], \\ 0, & \text{if } h \notin [h_{i,LB}, h_{i,UB}], \end{cases} \tag{3.27}$$

for every $i \in [1 : n]$ and $h \in [0 : 2n]$. The cost vector q defined in (3.27) provides a linearization of the objective function at integral points y such that $\|y - w^*\|_\infty \leq n$.

Theorem 3.3.4 (Solution of separable convex integer program)**.** *Suppose that* $\{f_i\}_{i=1}^n$ *are extended real-valued proper convex functions. Let* w^* *be an optimal solution to the linear relaxation of* (3.24)*, and let* $h_{i,LB}$*,* $h_{i,UB}$*, and* $q_{i,h}$ *be the constants defined previously. Then,* *the separable convex integer program* (3.24) *can be reformulated as follows:*

$$\min_{y,\delta} \sum_{i=1}^{n} \sum_{h=0}^{2n} q_{i,h} \cdot \delta_{i,h} \tag{3.28a}$$

$$\text{s.t. } Ay \geq b, \tag{3.28b}$$

$$y_i - \sum_{h=0}^{2n} \delta_{i,h} = \lfloor w_i^* \rfloor - n - 1, \qquad i \in [1:n], \tag{3.28c}$$

$$\delta_{i,h} = 1, \qquad h \in [0:h_{i,LB}],\ i \in [1:n], \tag{3.28d}$$

$$\delta_{i,h} = 0, \qquad h \in [h_{i,UB}+1:2n],\ i \in [1:n], \tag{3.28e}$$

$$y_i \in \mathbb{Z}_+,\ \delta_{i,h} \in \{0,1\}, \qquad h \in [0:2n],\ i \in [1:n]. \tag{3.28f}$$

In particular, an optimal solution (y^*, δ^*) *to* (3.28) *exists if and only if an optimal solution to* (3.24) *exists. The constraint matrix of the integer program* (3.28) *is totally unimodular, therefore, it is sufficient to solve the linear relaxation of* (3.28).

Proof of Theorem 3.3.4. First, note that constraints (3.28c) and (3.28f) imply that any solution $y \in \mathbb{Z}^n$ is such that $\|y - w^*\|_\infty \leq n$, where the infinite norm is defined as $\|a\|_\infty = \max_{i\in[1:n]} |a_i|$. By definition of the $q_{i,h}$, we note that

$$f_i\left(\lfloor w_i^* \rfloor - n + h\right) = \sum_{h=0}^{k} q_{i,h}, \tag{3.29}$$

for each $h \in [h_{i,LB}, h_{i,UB}]$ and $i \in [1:n]$. Because f_i is convex and univariate, the slopes of f_i are non-decreasing functions, so the sequence $\{q_{i,h}\}$ is non-decreasing over $h \in [h_{i,LB}+1 : h_{i,UB}]$, for every $i \in [1:n]$. This proves that among all possible representations of $f_i\left(\lfloor w_i^* \rfloor - n + h\right)$ as the binary variable encoding $\sum_{h=0}^{2n} q_{i,h} \cdot \delta_{i,h}$ the one with least objective cost is the right-hand side of (3.29). Thus, the formulation (3.28) is equivalent to

$$\min_{y} \quad \sum_{i=1}^{n} f_i(y_i) \tag{3.30a}$$

$$\text{s.t.} \quad \|y - w^*\|_\infty \leq n, \tag{3.30b}$$

$$\text{(3.24b)} - \text{(3.24c)}, \tag{3.30c}$$

and we know from Theorem 3.3.3 that an optimal solution to (3.30a) is also optimal to (3.24).

Recall that the constraint matrix A defined by the constraint (3.24b) is totally unimodular (TU), and by appending any canonical vector to columns or rows of a TU matrix, we preserve the TU property. Since the constraint matrix formed by (3.28b) and (3.28c) can be represented as

$$\widetilde{A} = \begin{array}{c} \begin{array}{cccc} \overline{\eta} & \delta_0 & \cdots & \delta_{2n} \end{array} \\ \begin{bmatrix} A & 0 & \cdots & 0 \\ I & -I & \cdots & -I \end{bmatrix} \end{array}, \tag{3.31}$$

where $\delta_h := (\delta_{i,h})_{i=1}^n$, for all $h \in [0:2n]$, we conclude that $\widetilde{A}$ is also TU. It is straightforward to see that all the other constraints coefficients when appended to $\widetilde{A}$ preserves the TU property. □

Note that Theorem 3.3.4 contains the statement of Theorem 3.3.1 as a particular case.

Theorem 3.3.1. *The separable convex integer program* (3.1) *can be reformulated as the* *following integer linear program:*

$$\min_{\overline{\eta},\delta} \sum_{\theta\in\Theta} \left(c_{\overline{\eta}}^{\theta} \cdot \overline{\eta}^{\theta} + \sum_{h=0}^{2|\Theta|} q_{\delta,h}^{\theta} \cdot \delta_h^{\theta} \right) \tag{3.6a}$$

$$\text{s.t. } \overline{\eta}^{\theta-1} \le \overline{\eta}^{\theta}, \qquad \overline{\eta}_{LB}^{\theta} \le \overline{\eta}^{\theta} \le \overline{\eta}_{UB}^{\theta}, \qquad \theta\in\Theta, \tag{3.6b}$$

$$\overline{\eta}^{\theta} - \sum_{h=0}^{2|\Theta|} \delta_h^{\theta} = \lfloor \overline{\eta}_{*,LR}^{\theta} \rfloor - |\Theta| - 1, \qquad \theta\in\Theta, \tag{3.6c}$$

$$\delta_h^{\theta} = 1, \qquad h \in [0 : h_{LB}^{\theta}],\ \theta\in\Theta, \tag{3.6d}$$

$$\delta_h^{\theta} = 0, \qquad h \in [h_{UB}^{\theta} + 1 : 2|\Theta|],\ \theta\in\Theta, \tag{3.6e}$$

$$\overline{\eta}^{\theta} \in \mathbb{Z}_+,\ \delta_h^{\theta} \in \{0,1\}, \qquad h = 0,\dots,2|\Theta|,\ \theta\in\Theta. \tag{3.6f}$$

In particular, an optimal solution $(\overline{\eta}_*, \delta_*)$ *to* (3.6) *exists if and only if an optimal solution to* (3.1) *exists. The constraint matrix induced by* (3.6b)-(3.6f) *is totally unimodular.*

Proof of Theorem 3.3.1. The proof is a direct application of Theorem 3.3.4. □

Thus, Theorem 3.3.1 guarantees the correctness of Algorithm 1.

3.3.3 Problem Size and Polynomial Solvability

In this section we prove the polynomial-time complexity from Theorem 3.3.2. We introduce the size of a mixed-integer linear program and bound the size of all the intermediate linear programs from Algorithm 1. We use the interior point algorithm from [12] to quantify the number of arithmetic operations necessary to find an optimal basic feasible solution of a bounded linear program. We conclude the time complexity from Theorem 3.3.2 by counting the number of required optimal basic feasible solutions.

Consider a mixed-integer linear program with integral coefficients:

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b, \\ & x \in \mathbb{R}^{n-k} \times \mathbb{Z}^k, \end{aligned} \tag{3.32}$$

where $c \in \mathbb{Z}^n$, $A \in \mathbb{Z}^{m \times n}$, and $b \in \mathbb{Z}^m$. All our integer program formulations of this chapter have integral coefficient matrices and the right-hand side vectors. The objective coefficients can be converted to integral numbers if one multiplies the denominator of each rational coefficient by the least common multiple among all denominators.

Suppose the feasible set $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is a non-empty polytope. Let Δ be the largest absolute value of the determinant of a submatrix of A. We define the size of (3.32) as

$$\mathbb{L} = \log_2(\Delta + 1) + \log_2\left(\max_{j \in [n]} |c_j| + 1\right) + \log_2\left(\max_{i \in [m]} |b_i| + 1\right) + \log_2(m + n). \tag{3.33}$$

The following lemma provides a bound for any basic feasible solution and associated objective cost $c^\top x^*$ using the size $\mathbb{L}$.

Lemma 3.3.4. *Any basic feasible solution x^* to* (3.32) *and associated objective cost $c^\top x^*$ have upper bounds*

$$\|x^*\|_\infty \leq 2^{\mathbb{L}}, \qquad |c^\top x^*| \leq 2^{2\mathbb{L}}. \tag{3.34}$$

Proof of Lemma 3.3.4. If x^* is a basic feasible solution then it is a solution to the linear system $Hx = g$, where $H \in \mathbb{Z}^{n \times n}$ is a nonsingular submatrix of A and $g \in \mathbb{Z}^n$ is a subvector of b. From the Cramer's rule, we have that

$$x_j = \frac{\det H_j}{\det H},$$

where H_j is the matrix formed by replacing the j-th column of H by g. We bound the determinant of H_j by using the Cofactor expansion formula for the j-th column of H. Indeed, $\det H_j = \sum_{i=1}^{n}(-1)^{i+j} g_i \det H_j^i$, where H_j^i is the matrix obtained from H_j by eliminating the i-th row and j-th column. Thus,

$$|\det H_j| \leq \sum_{i=1}^{n} |g_i||\det H_j^i| \leq \sum_{i=1}^{n} |g_i|\Delta \leq 2^{\left(\log_2(\Delta+1)+\log_2(\|g\|_\infty+1)+\log_2(n)\right)} \leq 2^{\mathbb{L}}.$$

Because the matrix H is integral and nonsingular, we have that $|\det H| \geq 1$. This implies that $|x_j| = |\det H_j|/|\det H| \leq 2^{\mathbb{L}}$ for all $j \in [n]$.

For the objective cost, we have that

$$|c^\top x| \leq \sum_{j=1}^{n} |c_j||x_j| \leq \|c\|_\infty \sum_{j=1}^{n} |x_j| \leq \|c\|_\infty n 2^{\mathbb{L}} \leq 2^{\left(\log_2(\|c\|_\infty+1)+\log_2(n)+\mathbb{L}\right)} \leq 2^{2\mathbb{L}}.$$

□

We now define the size of fleet sizing instance (3.4). First, we formulate (3.4) as an integer program by expanding the value function $\overline{F}_\theta(\overline{\eta}^\theta)$ with the reformulation described in Lemma 3.3.3:

$$\min_{\overline{\eta},\overline{\zeta}} \quad \sum_{\theta\in\Theta}\left[c_{\overline{\eta}}^\theta \cdot \overline{\eta}^\theta + \sum_{i=0}^{k-1} \overline{p}_{\zeta,i}^\theta \cdot \overline{\zeta}_i^\theta\right] \tag{3.35a}$$

$$\text{s.t.} \quad \overline{\eta}^{\theta-1} \leq \overline{\eta}^\theta, \qquad \overline{\eta}_{LB}^\theta \leq \overline{\eta}^\theta \leq \overline{\eta}_{UB}^\theta, \qquad \overline{\eta}^\theta \geq 0, \qquad \theta \in \Theta, \tag{3.35b}$$

$$\overline{\zeta}_{k-1}^\theta - \overline{\eta}^\theta = 0, \qquad \theta \in \Theta, \tag{3.35c}$$

$$\overline{\zeta}_i^\theta - \overline{\zeta}_{i-W}^\theta + \left(\left\lfloor \frac{W}{k} \right\rfloor + \mathbb{I}_{[i+1,\infty)}(W\%k)\right)\overline{\zeta}_{k-1}^\theta \geq \widetilde{d}_i^\theta, \qquad i \in [0:k-1],\ \theta \in \Theta, \tag{3.35d}$$

$$\overline{\zeta}_0^\theta \geq 0, \qquad \theta \in \Theta, \tag{3.35e}$$

$$\overline{\zeta}_i^\theta - \overline{\zeta}_{i-1}^\theta \geq 0, \qquad i \in [1:k-1],\ \theta \in \Theta, \tag{3.35f}$$

$$\overline{\eta}^\theta,\ \overline{\zeta}_i^\theta \in \mathbb{Z}, \qquad i \in [0:k-1],\ \theta \in \Theta. \tag{3.35g}$$

Note that the integer program (3.35) has $n = (k+1)|\Theta|$ variables and $m = (2k+5)|\Theta|$ linear constraints. Denote by Δ the maximum absolute value of the determinant of a submatrix of the constraint matrix in (3.35). Let $c = [c_{\overline{\eta}}, \overline{p}_{\zeta}]$ and $b = [\overline{\eta}_{LB}, \overline{\eta}_{UB}, \widetilde{d}]$ be the objective cost and right-hand side vectors, respectively. Then, the size of the integer program (3.35) is well-defined by the formula (3.33):

$$\begin{aligned}\mathbb{L} =& \log_2(\Delta+1) + \log_2\left(\max\left\{\|c_{\overline{\eta}}\|_\infty,\ \|\overline{p}_{\zeta}\|_\infty\right\}+1\right) \\ &+ \log_2\left(\max\left\{\|\overline{\eta}_{LB}\|_\infty,\ \|\overline{\eta}_{UB}\|_\infty,\ \|\widetilde{d}\|_\infty\right\}+1\right) \\ &+ \log_2\left(m+n\right).\end{aligned} \tag{3.36}$$

The feasible region of the linear relaxation of (3.35) is a polytope since all the variables are bounded. Indeed, it follows from (3.35b), (3.35c), (3.35e), and (3.35f) the following inequalities

$$0 \le \overline{\zeta}_0^{\theta} \le \overline{\zeta}_1^{\theta} \le \cdots \le \overline{\zeta}_{k-1}^{\theta} = \overline{\eta}^{\theta} \le \overline{\eta}_{UB}^{\theta}.$$

We are now in a position to prove the time complexity of Algorithm 1. First, we must guarantee that the sizes of all the intermediate linear programs are polynomially bounded by the size $\mathbb{L}$ of the integer program (3.35). Indeed, denote by $\mathbb{L}_{\theta,h}$ the size of the operational problem (3.18), and let $n_{\theta,h}$ and $m_{\theta,h}$ be the associated number of variables and constraints. Also, recall that the coefficient matrix of (3.18) is TU. Then,

$$\begin{aligned}\mathbb{L}_{\theta,h} =& 1 + \log_2\left(\|\overline{p}_{\zeta}^{\theta}\|_\infty + 1\right) \\ &+ \log_2\left(\max\left\{\|\widetilde{d}^{\theta}\|_\infty,\ \left|\lfloor\overline{\eta}_{*,LR}^{\theta}\rfloor - |\Theta| - 1 + h\right|\right\}+1\right) \\ &+ \log_2\left(m_{\theta,h}+n_{\theta,h}\right),\end{aligned} \tag{3.37}$$

where the number of variables is $n_{\theta,h} = k$ and the number of constraints is $m_{\theta,h} = 2k+1$. Similarly, denote by $\mathbb{L}_P$ the size of the proximity problem (3.6), and let n_P and m_P be the corresponding number of variables and constraints. Recall that the coefficient matrix of (3.6) is also TU. Then,

$$\begin{aligned}\mathbb{L}_P =& 1+\log_2\left(\max\left\{\|c_{\overline{\eta}}\|_\infty,\ \|q_\delta\|_\infty\right\}+1\right)\\&+\log_2\left(\max\left\{\|\overline{\eta}_{LB}\|_\infty,\ \|\overline{\eta}_{UB}\|_\infty,\ \|\lfloor\overline{\eta}_{*,LR}\rfloor-(|\Theta|+1)\cdot e\|_\infty\right\}+1\right)\\&+\log_2\left(m_P+n_P\right).\end{aligned} \tag{3.38}$$

where e is a vector of 1's, the number of variables of (3.6) is $n_P = 2|\Theta|^2+|\Theta|$, and the number of constraints is $m_P = 2|\Theta|^2+4|\Theta|+1$.

Lemma 3.3.5. *The linear program sizes $\mathbb{L}_{\theta,h}$ and $\mathbb{L}_P$ are linearly bounded by the size $\mathbb{L}$:*

$$\mathbb{L}_{\theta,h}\le 2\mathbb{L}+2,\qquad \mathbb{L}_P\le 8\mathbb{L}+13, \tag{3.39}$$

for all $\theta\in\Theta$ and $h=0,1,\dots,2|\Theta|$.

Proof of Lemma 3.3.5. From the definition of $\mathbb{L}$, we obtain the upper bound:

$$\mathbb{L}_{\theta,h}\le\mathbb{L}+\log_2\left(\max\left\{\|\widetilde{d}^\theta\|_\infty,\ \left|\lfloor\overline{\eta}^\theta_{*,LR}\rfloor-|\Theta|-1+h\right|\right\}+1\right). \tag{3.40}$$

Because $(\overline{\eta}_{*,LR},\overline{\zeta}_{*,LR})$ is an optimal basic feasible solution, we know from Lemma 3.3.4 that $0\le\lfloor\overline{\eta}^\theta_{*,LR}\rfloor\le\overline{\eta}^\theta_{*,LR}\le 2^{\mathbb{L}}$. This implies that

$$\left|\lfloor\overline{\eta}^\theta_{*,LR}\rfloor-|\Theta|-1+h\right|\le 2^{\mathbb{L}}+\left|h-|\Theta|-1\right|\le 2^{\mathbb{L}+1} \tag{3.41a}$$

$$\implies \mathbb{L}_{\theta,h}\le\mathbb{L}+\log_2(2^{\mathbb{L}+1}+1)\le 2\mathbb{L}+2, \tag{3.41b}$$

for all $\theta\in\Theta$ and $h=0,1,\dots,2|\Theta|$.

It follows from (3.36) and Lemma 3.3.4 the upper bound $|\widetilde{F}_\theta(\eta)|\le 2^{2\mathbb{L}_{\theta,h}}$. Then, we use this inequality and (3.41b) to get $|q^\theta_{\delta,h}|\le 2^{2\mathbb{L}_{\theta,h}+1}\le 2^{4\mathbb{L}+5}$. Hence, we have the following upper bound for $\mathbb{L}_P$:

$$\mathbb{L}_P \leq 1 + \log_2\left(2^{4\mathbb{L}+5} + 1\right) + \log_2\left(2^{2\mathbb{L}+1} + 1\right) + \log_2(m_P + n_P) \tag{3.42a}$$

$$\leq (6\mathbb{L} + 9) + \log_2(m_P + n_P) \tag{3.42b}$$

$$= (6\mathbb{L} + 9) + \log_2(4|\Theta|^2 + 5|\Theta| + 1) \tag{3.42c}$$

$$\leq (6\mathbb{L} + 9) + \log_2\left(16 \cdot |\Theta|^2\right) \tag{3.42d}$$

$$= (6\mathbb{L} + 13) + 2\log_2(|\Theta|) \tag{3.42e}$$

$$\leq 8\mathbb{L} + 13. \tag{3.42f}$$

□

We use the time-complexity $O(((m+n)n^2 + (m+n)^{1.5}n)\mathbb{L})$ of the interior point algorithm from [12] as the complexity to find an optimal basic feasible solutions for a linear programs instance, where $\mathbb{L}$ is the size of the instance, m is the number of constraints, and n is the number of variables. Note that if the number of constraints is $m = O(n)$ the time complexity reduces to $O(n^3\mathbb{L})$.

Theorem 3.3.2. *The time complexity to obtain an optimal integral solution η^* to* (3.1) *using the Algorithm 1 is $O\big((k^3|\Theta|^3 + |\Theta|^6)\mathbb{L}\big)$ arithmetic operations at a precision of $O(\mathbb{L})$ bits.*

Proof. The number of variables n and constraints m of (3.35) is $O(k|\Theta|)$. So, the time complexity to compute an optimal basic feasible solution $(\overline{\eta}_{*,LR}, \overline{\zeta}_{*,LR})$ to the linear relaxation of (3.35) is $O\big((k|\Theta|)^3\mathbb{L}\big)$.

Since the number of variables $n_{\theta,h}$ and constraints $m_{\theta,h}$ of each subproblem (3.18) is $O(k)$ the time complexity to solve each of them is $O(k^3\mathbb{L}_{\theta,h})$. By Lemma 3.3.5, we have that $\mathbb{L}_{\theta,h} = O(\mathbb{L})$ and this implies that the time complexity to solve each instance of (3.18) is indeed $O(k^3\mathbb{L})$. Hence, the time complexity to find the coefficient q_δ of the proximity problem (3.6) is $O(k^3|\Theta|^2\mathbb{L})$.

Finally, one needs to solve the proximity problem (3.6) which has $O(|\Theta|^2)$ variables and constraints. This implies a time complexity of $O(|\Theta|^6\mathbb{L}_P)$, and again by Lemma 3.3.5, we can replace the size $\mathbb{L}_p$ by $\mathbb{L}$ in the time complexity estimate. Therefore, the total time complexity of Algorithm 1 is

$$O(k^3|\Theta|^3\mathbb{L}) + O(k^3|\Theta|^2\mathbb{L}) + O(|\Theta|^6\mathbb{L}) = O\big((k^3|\Theta|^3 + |\Theta|^6)\mathbb{L}\big). \tag{3.43}$$

□

3.4 A special class of polynomially solvable Separable Integer Programs

We have described in Section 3.3 the main ingredients for the polynomial solvability of our Fleet Sizing problem (3.1). Recall that besides the separability structure of the objective function and the TU property, we also needed the tight convex lower approximation induced by the linear relaxation of the operational problem (3.2). Those two properties were crucial for the proof of the polynomial-time complexity of Algorithm 1.

The goal of this section is to describe a more general structure of the constraint matrix of the second stage problem that guarantees the integrality of the associated linear relaxation for every argument $y_i \in \mathbb{Z}$. Here, y_i is the argument of the value function $f_i(y_i)$ that defines the separable objective function $\sum_{i=1}^{k} f_i(y_i)$ from the first stage. This class of Separable Integer Program with TU constraints is still be polynomially solvable using Algorithm 1.

3.4.1 Value function induced by a Dyad Contiguous Row (DCR) matrix

We say that a matrix A is *Dyad Contiguous Row* (DCR) matrix if A is an integral matrix such that each row contains a block of consecutive θ's followed by a block of consecutive $\theta - 1$'s or $\theta + 1$'s, for any $\theta \in \mathbb{Z}$. The first and last entries of each row are also considered consecutive. The length of each block may vary across the rows but the total number of blocks must be less than or equal to 2. This contains the definition of [6] of a row-circular matrix.

Below is an example of a DCR matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix}. \tag{3.44}$$

It is instructive to note that the coefficient matrix of the operational problem formulation (3.7) is a DCR matrix.

Consider the following pure integer program:

$$\begin{aligned} f(y) = \min \quad & c^\top \zeta && (3.45a) \\ \text{s.t.} \quad & A\zeta \leq b, && (3.45b) \\ & \zeta_1 = y, && (3.45c) \\ & \zeta \in \mathbb{Z}_+^k, && (3.45d) \end{aligned}$$

where A is a DCR matrix. Then, we have the following Proposition.

Proposition 3.4.1. *The linear relaxation of* (3.45) *induces a tight convex lower approximation of* $f(y)$.

Proof of Proposition 3.4.1. It is enough to prove that the polyhedron $P(y)$ defined by the linear constraints of (3.45) is integral for every $y \in \mathbb{Z}$. Indeed, consider the symmetry breaking unimodular transformation $(R\zeta)_i = \sum_{l=1}^{i} \zeta_l$ from (3.16). Its inverse is given by the formula $(R^{-1}\overline{\zeta})_1 = \overline{\zeta}_1$ and $(R^{-1}\overline{\zeta})_i = \overline{\zeta}_i - \overline{\zeta}_{i-1}$, if $2 \leq i \leq k$.

By applying the change of variables $\overline{\zeta} = R\zeta$, we obtain the reformulation:

$$\begin{aligned} f(y) = \min \quad & c^\top \zeta && (3.46a) \\ \text{s.t.} \quad & \overline{A}_{[2:k]}\overline{\zeta}_{[2:k]} + a_1\overline{\zeta}_1 \leq b, && (3.46b) \end{aligned}$$

$$\overline{\zeta}_1 = y, \tag{3.46c}$$

$$R^{-1}\overline{\zeta} \geq 0, \tag{3.46d}$$

$$\overline{\zeta} \in \mathbb{Z}^k, \tag{3.46e}$$

where $\overline{\zeta}_{[2:k]}$ is the subvector of $\overline{\zeta}$ defined by the coordinates from 2 to k, $\overline{A}_{[2:k]}$ is the matrix induced by the columns of $\overline{A}$ from 2 to k, $\overline{A}$ denotes the matrix AR^{-1}, and a_i denotes the i-th column of A. Note that the first column of $\overline{A}$ is a_1 and the i-th column of $\overline{A}$ is the difference $a_i - a_{i-1}$, if i is greater than or equal to 2.

It follows from the DCR property of A that each row of $\overline{A}_{[2:k]}$ has at most one $+1$ and -1. The same property holds for the inverse R^{-1}. If we replace $\overline{\zeta}_1$ by the constant y we obtain an equivalent formulation with totally unimodular constraint. This concludes that the polyhedron $P(y)$ is integral. □

CHAPTER 4

STATIONARY INFINITE-DIMENSIONAL LINEAR PROGRAMS.

4.1 Introduction

In this chapter, we introduce a notion of primal and dual for infinite-dimensional *stationary* linear programs based on a restriction to ℓ_∞ and ℓ_1 spaces of appropriate dimensions. We motivate this approach from the fixed-point formulation of discounted stationary programs and also prove weak duality. Furthermore, we illustrate with a hydro-thermal infinite-dimensional program and a variation of the [49] example that strong duality may hold for a large class of infinite-dimensional stationary linear programs. Weak duality even fails for the later example if we remove the ℓ_∞ and ℓ_1 constraints. We also show that the value function of the hydro-thermal infinite-dimensional stationary linear program is piecewise linear convex with countably many affine functions when the state variable upper bound is $+\infty$.

4.2 Related work

Duality in infinite-dimensional linear programs has been widely studied in operations research. The work of [49] presents an example of an infinite-dimensional dual program obtained by the finite-dimensional duality rules for which weak duality does not hold. Such a dual problem is referred to as the *natural dual*.

[50] provides a review for duality in abstract topological vector spaces and presents the Slater condition as a sufficient condition for strong duality. This dual notion is called the *algebraic dual*. The book [51] contains examples and applications of algebraic duality theory but in general it is not straightforward to compute the algebraic dual of a given infinite-dimensional linear program. Some recent developments and extensions of the Algebraic

Duality Theory can be found in [52] and [53]. The work of [54] presents sufficient conditions different than the Slater condition on the algebraic dual of countably infinite linear programs for both weak and strong duality to hold. [8] consider finite-dimensional dual approximations as a way to establish properties for the natural dual which avoids the necessity of the closedness condition for the primal problem and the Slater for the dual. Moreover, in order to prove strong duality, [8] assume a condition called transversality, which is a convergence to zero condition of a sequence of dual optimal solutions. [55] further specialize this strong duality result and the transversality condition for the case of finitely many variables for each constraint. However, the transversality condition is hard to verify in practice. [56] characterize the value function of an infinite-horizon single-item lot-sizing problem and show that it has all the properties of a finite-dimensional mixed-integer value function. That is, they show that the value function is piecewise linear, lower-semicontinuous, and sub-additive. Finally, [57] model an infinite-horizon stationary stochastic program using the Bellman equation for the natural dual problem. They develop a cutting plane type method to solve it.

4.3 Contributions

1. **A duality framework:** Based on the duality rules for finite dimensional LPs and restrictions to appropriate ℓ_∞ and ℓ_1 spaces, we introduce primal and dual infinite dimensional stationary linear programs and prove weak duality.

2. **Evidence of a strong duality result:** Strong duality may hold for a large class of problems in our infinite dimensional setting as illustrated by a stationary hydrothermal power generation planning problem. Using a counter-example, we show that weak duality may fail if one disregards the ℓ_∞ and ℓ_1 set constraints. However, that same example satisfies strong duality when the ℓ_∞ and ℓ_1 constraints are enforced.

4.4 Weak Duality for infinite-dimensional stationary linear programs

Let $A, B \in \mathbb{R}^{m \times n}$ be general coefficient matrices, let $c \in \mathbb{R}^n$ be the unit cost associated to the decision variable, and let $b \in \mathbb{R}^m$ be the right-hand side vector of the stationary constraints. Our decision vector is denoted by $x_1 \in \mathbb{R}^n$, the initial state is $x_0 \in \mathbb{R}^n$, and $\alpha \in (0,1)$ is our discount factor. We present below the fixed-point formulation of our infinite-horizon discounted stationary program:

$$V_F(x_0) := \inf \quad c^\top x_1 + \alpha \cdot V_F(x_1) \tag{4.1a}$$

$$\text{s.t.} \quad Ax_1 \geq b - Bx_0, \tag{4.1b}$$

$$x_1 \in \mathbb{R}^n. \tag{4.1c}$$

A solution to (4.1) is called a fixed-point value function V_F.

Note that (4.1) is written with an infimum since we do not know the shape of V_F and there is no guarantee that the optimal value can be attained by a given feasible solution x_1. Observe also that the fixed-point problem (4.1) for extended real-valued functions can have multiple solutions if no additional regularity condition is imposed for V_F since $V_F \equiv +\infty$ or $V_F \equiv -\infty$ are solutions to (4.1), and other similar solutions can also be constructed.

Another approach for (4.1) is to formulate it as an infinite-dimensional *stationary* linear program by recursively expanding the value function V_F using the associated linear program. The problem with this approach is that the resulting objective function $\sum_{t=1}^{\infty} \alpha^{t-1} c^\top x_t$ may not be a convergent series. [7] modifies the definition of the objective function by taking the $\liminf$ of such series, that is, $\liminf_{T \in \mathbb{Z}_+} \sum_{t=1}^{T} \alpha^{t-1} c^\top x_t$. However, this latter alternative breaks the linear structure of the objective function. Another possibility is to assume some stationary bound on x_t, making the series $\sum_{t=1}^{\infty} \alpha^{t-1} c^\top x_t$ absolutely convergent. However, this approach is problem dependent and it can be challenging to find uniform bounds in general.

We propose a simple solution for the formulation of an infinite-dimensional *stationary* linear program. We consider the additional constraint that the entire sequence of decisions $\boldsymbol{x} := (x_1, x_2, x_3, \ldots)$ is $\ell_\infty(\mathbb{R}^n)$-bounded and denote it as $\boldsymbol{x} \in \ell_\infty(\mathbb{R}^n)$. That is, we assume that the supremum norm $\|\boldsymbol{x}\|_\infty := \sup_{t\geq 1} \|x_t\|$ is finite for any feasible sequence $\boldsymbol{x}$, where $\|\cdot\|$ is any norm in $\mathbb{R}^n$. Thus, our primal infinite-dimensional stationary linear program is defined as

$$V(x_0) := \inf \quad \sum_{t=1}^{\infty} \alpha^{t-1} c^\top x_t \tag{4.2a}$$

$$\text{s.t.} \quad Ax_t + Bx_{t-1} \geq b, \qquad t \geq 1, \tag{4.2b}$$

$$x_t \in \mathbb{R}^n,\ \boldsymbol{x} \in \ell_\infty(\mathbb{R}^n), \quad t \geq 1. \tag{4.2c}$$

Note that the set constraint $\boldsymbol{x} \in \ell_\infty(\mathbb{R}^n)$ equivalently means that there exists $r \geq 0$ so that $\|x_t\| \leq r$ for all $t \geq 1$. The stationary bound approach assumes that $\|\boldsymbol{x}\|_\infty \leq r$ for some fixed r, which is more restrictive than the $\ell_\infty(\mathbb{R}^n)$ constraint. In the finite-dimensional setting, this difference in formulations corresponds to the difference between a program with a norm ball constraint and an unconstrained program.

By convention, we define the value function V at x_o as $+\infty$ if the feasible set for (4.2) is empty. We will see in section 4.5.2 examples of amplifications effects caused by the constraints' relations which result in the problem (4.2) being infeasible or having just one feasible solution.

Lemma 4.4.1. *The value function $V : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ defined in* (4.2) *is a fixed-point value function for* (4.1).

Proof. The goal is to show that the optimal value of

$$\inf \quad c^\top x_1 + \alpha \cdot V(x_1) \tag{4.3a}$$

$$\text{s.t.} \quad Ax_1 \geq b - Bx_0, \tag{4.3b}$$

$$x_1 \in \mathbb{R}^n. \tag{4.3c}$$

is $V(x_0)$ for every $x_0 \in \mathbb{R}^n$. Indeed, we can replace the definition of $V(x_1)$ from (4.2), enumerate the sequence of decisions x_t starting from $t = 2$ onward, and group the infimum operators to get the following reformulation of (4.3):

$$
\begin{aligned}
\inf \quad & c^\top x_1 + \alpha \cdot \sum_{t=2}^{\infty} \alpha^{t-2} c^\top x_t && \text{(4.4a)} \\
\text{s.t.} \quad & Ax_1 \geq b - Bx_0, && \text{(4.4b)} \\
& Ax_t + Bx_{t-1} \geq b, \qquad t \geq 2, && \text{(4.4c)} \\
& x_1 \in \mathbb{R}^n, \ \{x_t\}_{t=2}^{\infty} \in \ell_\infty(\mathbb{R}^n). && \text{(4.4d)}
\end{aligned}
$$

The result then follows by noting that (4.4) is just an equivalent representation of (4.2).

□

The advantage of this approach is that we can define a notion of dual problem using a similar technique. We define a dual problem based on the same duality rules for finite dimensional linear programs but include $\ell_1(\mathbb{R}^n)$ as an additional set constraint. We say that the dual sequence $\boldsymbol{\mu} := \{\mu_t\}_{t=1}^{\infty}$ belongs to $\ell_1(\mathbb{R}^m)$ if and only if the ℓ_1-norm $\|\boldsymbol{\mu}\|_1 := \sum_{t=1}^{\infty} \|\mu_t\|$ is finite, where $\|\cdot\|$ is any finite dimensional norm in $\mathbb{R}^m$. Below is our definition of dual problem:

$$
\begin{aligned}
W(x_0) := \sup \quad & -(Bx_0)^\top \mu_1 + \sum_{t=1}^{\infty} b^\top \mu_t && \text{(4.5a)} \\
\text{s.t.} \quad & A^\top \mu_t + B^\top \mu_{t+1} = \alpha^{t-1} c, \quad t \geq 1, && \text{(4.5b)} \\
& \mu_t \geq 0, \ \boldsymbol{\mu} \in \ell_1(\mathbb{R}^m), \qquad t \geq 1. && \text{(4.5c)}
\end{aligned}
$$

Note that the series $\sum_{t=1}^{\infty} b^\top \mu_t$ is absolutely convergent since $\boldsymbol{\mu}$ belongs to $\ell_1(\mathbb{R}^m)$. The following Theorem proves that weak duality holds between (4.2) and (4.5).

Theorem 4.4.1 (Weak Duality)**.** *Problems* (4.2) *and* (4.5) *satisfy weak duality. That is,*

$$W(x_0) \leq V(x_0), \tag{4.6}$$

for every $x_0 \in \mathbb{R}^n$.

Proof. Let $\boldsymbol{x}$ and $\boldsymbol{\mu}$ be any primal and dual feasible solution to (4.2) and (4.5), respectively. Then, using the relations of the primal and dual constraints we get the following identities:

$$\sum_{t=1}^{\infty} \alpha^{t-1} c^\top x_t = \sum_{t=1}^{\infty} \left[A^\top \mu_t + B^\top \mu_{t+1} \right]^\top x_t = \sum_{t=1}^{\infty} \mu_t^\top A x_t + \mu_{t+1}^\top B x_t. \tag{4.7}$$

Because the primal sequence $\boldsymbol{x}$ belongs to $\ell_\infty(\mathbb{R}^n)$ and the dual sequence $\boldsymbol{\mu}$ belongs to $\ell_1(\mathbb{R}^m)$, both series $\sum_{t=1}^{\infty} \mu_t^\top A x_t$ and $\sum_{t=1}^{\infty} \mu_{t+1}^\top B x_t$ are both absolutely convergent, and hence the following identity holds:

$$\sum_{t=1}^{\infty} \mu_t^\top A x_t + \mu_{t+1}^\top B x_t = \sum_{t=1}^{\infty} \mu_t^\top A x_t + \sum_{t=1}^{\infty} \mu_{t+1}^\top B x_t. \tag{4.8}$$

We then replace (4.8) in (4.7) and conclude the duality argument with a standard algebraic manipulation:

$$\begin{aligned}\sum_{t=1}^{\infty} \alpha^{t-1} c^\top x_t &= \sum_{t=1}^{\infty} \mu_t^\top A x_t + \sum_{t=2}^{\infty} \mu_t^\top B x_{t-1} \\ &= -(Bx_0)^\top \mu_1 + \sum_{t=1}^{\infty} \Big[\underbrace{A x_t + B x_{t-1}}_{\geq b} \Big]^\top \underbrace{\mu_t}_{\geq 0} \geq -(Bx_0)^\top \mu_1 + \sum_{t=1}^{\infty} b^\top \mu_t.\end{aligned}$$

□

4.4.1 Rescaled dual problem

Another candidate to dual problem is obtained from (4.5) by rescaling the dual variables through the *bijective* linear transformation $\mu_t \to \mu_t/\alpha^{t-1}$ and imposing that $\boldsymbol{\mu}$ is $\ell_\infty(\mathbb{R}^m)$-bounded:

$$W_R(x_0) := \sup \quad -(Bx_0)^\top \mu_1 + \sum_{t=1}^{\infty} \alpha^{t-1} b^\top \mu_t \tag{4.9a}$$

$$\text{s.t.} \quad A^\top \mu_t + \alpha B^\top \mu_{t+1} = c, \qquad t \geq 1, \tag{4.9b}$$

$$\mu_t \geq 0,\ \boldsymbol{\mu} \in \ell_\infty(\mathbb{R}^m), \qquad t \geq 1. \tag{4.9c}$$

Note that the rescaled dual problem (4.9) is an infinite-dimensional stationary linear program equivalent to (4.2), except for the first objective term $-(Bx_0)^\top \mu_1$. Next proposition proves that (4.9) is a more constrained dual.

Proposition 4.4.1. *The feasible region from the rescaled dual problem* (4.9) *is smaller than or equal to the feasible region of the original dual problem* (4.5)*. Moreover, the inequality* $W_R(x_0) \leq W(x_0)$ *holds for all* $x_0 \in \mathbb{R}^n$.

Proof. Indeed, given a feasible solution $\boldsymbol{\mu}$ to (4.9), the solution $\widetilde{\boldsymbol{\mu}}$ defined as $\widetilde{\mu}_t = \alpha^{t-1}\mu_t$ is feasible to (4.5) with the same objective value. It follows then that $\widetilde{\boldsymbol{\mu}}$ is $\ell_1(\mathbb{R}^m)$-bounded since

$$\sum_{t=1}^{\infty} \|\widetilde{\mu}_t\| = \sum_{t=1}^{\infty} \alpha^{t-1}\|\mu_t\| \leq \|\boldsymbol{\mu}\|_\infty \cdot \sum_{t=1}^{\infty} \alpha^{t-1} = \frac{\|\boldsymbol{\mu}\|_\infty}{1-\alpha} < +\infty.$$

On the other hand, the converse is not necessarily true. See the example below. □

For instance, consider the dual and the rescaled dual problems

$$\begin{aligned} \sup\ & \sum_{t=1}^{\infty} -\widetilde{\mu}_t \\ \text{s.t. }\ & \widetilde{\mu}_t \geq \alpha^{t-1}, \quad t \geq 1, \\ & \widetilde{\boldsymbol{\mu}} \in \ell_1(\mathbb{R}), \end{aligned} \qquad \text{and} \qquad \begin{aligned} \sup\ & \sum_{t=1}^{\infty} -\alpha^{t-1}\mu_t && (4.10\text{a}) \\ \text{s.t. }\ & \mu_t \geq 1, \qquad t \geq 1, && (4.10\text{b}) \\ & \boldsymbol{\mu} \in \ell_\infty(\mathbb{R}). && (4.10\text{c}) \end{aligned}$$

Given $\beta \in (0,1)$ greater than α, the dual sequence $\widetilde{\boldsymbol{\mu}}$ defined as $\widetilde{\mu}_t = \beta^{t-1}$ for all $t \geq 1$ is dual feasible. However, the image $\boldsymbol{\mu}$ defined by the rescaling map is not feasible for the rescaled dual problem since $\boldsymbol{\mu}$ is not $\ell_\infty(\mathbb{R})$-bounded:

$$\sup_{t \geq 1} |\mu_t| = \sup_{t \geq 1} \left(\frac{\beta}{\alpha}\right)^{t-1} = +\infty.$$

Hence, the rescaled dual problem may have a strictly smaller feasible set.

4.5 Examples

4.5.1 An infinite-horizon Hydro-Thermal Power Planning problem

The focus of this section is to characterize the value function of an infinite-horizon stationary discounted hydro-thermal power planning problem as well as to present an optimal solution for such a program. For this problem, we only assume that there is only one hydro plant and one thermal plant.

The hydro plant has an energy reservoir with maximum storage capacity $\overline{v}$ and a constant energy inflow a from one stage to another. The hydro plant can output energy at most equal to its stored energy v_0 plus the energy inflow a. The thermal plant instead has an arbitrary generation capacity. Both the thermal and the hydro generation must add up to the energy demand d. The thermal generation unit cost for our problem is c while the associated cost for hydro generation is 0. We denote the discount factor of this problem by α, which is a positive number less than 1.

The decision variables of the infinite-horizon stationary discounted hydro-thermal problem are the final stored energy v_1, the hydro generation q_1, and the thermal generation g_1. Our fixed-point formulation is written below:

$$
\begin{aligned}
V_F(v_0) = \inf \quad & cg_1 + \alpha \cdot V_F(v_1) && \text{(4.11a)}\\
\text{s.t.} \quad & v_1 + q_1 = a + v_0, && \text{(4.11b)}\\
& g_1 + q_1 = d, && \text{(4.11c)}\\
& v_1 \leq \overline{v}, && \text{(4.11d)}\\
& v_1, g_1, q_1 \geq 0. && \text{(4.11e)}
\end{aligned}
$$

We assume that all parameters c, a, d, and $\overline{v}$ are non-negative scalars and that the energy demand d is greater than the energy inflow a, i.e., $d > a$.

We thus find a fixed-point value function V_F and the corresponding optimal solutions of (4.11) by defining the infinite-dimensional stationary linear program as in (4.2) and the corresponding dual problem as in (4.5). Indeed, the infinite-dimensional stationary linear program counterpart of (4.11) is given by

$$
\begin{aligned}
V(v_0) := \inf \quad & \sum_{t=1}^{\infty} \alpha^{t-1} \cdot c g_t && && (4.12a)\\
\text{s.t.} \quad & v_t - v_{t-1} + q_t = a, && t \geq 1, && (4.12b)\\
& g_t + q_t = d, && t \geq 1, && (4.12c)\\
& v_t \leq \overline{v}, && t \geq 1, && (4.12d)\\
& v_t, g_t, q_t \geq 0, && t \geq 1, && (4.12e)\\
& \boldsymbol{v}, \boldsymbol{g}, \boldsymbol{q} \in \ell_\infty(\mathbb{R}). && && (4.12f)
\end{aligned}
$$

We note that all the variables have explicit or implicit uniform bounds. For instance, all the variables are non-negative, both the thermal g_t and the hydro generation q_t are upper bounded by the energy demand d, and the stored energy v_t is uniformly bounded by $v_0 + a$ for every $t \geq 1$. Thus, the $\ell_\infty(\mathbb{R})$ set constraint in (4.12) is redundant for $\boldsymbol{v}$, $\boldsymbol{g}$, and $\boldsymbol{q}$.

Following the same idea from the general dual case (4.5), we define the dual problem of (4.12):

$$
\begin{aligned}
W(v_0) := \sup \quad & v_0 \mu_1 + a \cdot \sum_{t=1}^{\infty} \mu_t + d \cdot \sum_{t=1}^{\infty} \gamma_t + \overline{v} \cdot \sum_{t=1}^{\infty} u_t && && (4.13a)\\
\text{s.t.} \quad & \mu_t - \mu_{t+1} + u_t \leq 0, && t \geq 1, && (4.13b)\\
& \mu_t + \gamma_t \leq 0, && t \geq 1, && (4.13c)\\
& \gamma_t \leq \alpha^{t-1} c, && t \geq 1, && (4.13d)\\
& \mu_t, \gamma_t \in \mathbb{R},\ u_t \leq 0, && t \geq 1, && (4.13e)\\
& \boldsymbol{\mu}, \boldsymbol{\gamma}, \boldsymbol{u} \in \ell_1(\mathbb{R}). && && (4.13f)
\end{aligned}
$$

Note that weak duality follows from Lemma 4.4.1.

Proposition 4.5.1 (Strong Duality for the hydro-thermal problem)**.** *If* $v_0 > d + \overline{v} - a$*, then the primal problem* (4.12) *is infeasible and the dual* (4.13) *problem is unbounded. If* $v_0 \leq d + \overline{v} - a$*, then the primal and dual problems satisfy strong duality. In particular, the primal and dual optimal values are equal to*

$$V(v_0) = W(v_0) = \begin{cases} \alpha^{i-1}c \cdot \left[\frac{d-a}{(1-\alpha)} - (v_0 - \theta_{i-1})\right], & \text{if } v_0 \in [\theta_{i-1}, \theta_i] \cap [0, d + \overline{v} - a], \\ +\infty, & \text{otherwise,} \end{cases} \tag{4.14}$$

where θ_i *denotes* $i(d-a)$ *for all* $i \geq 1$*. The primal and dual optimal solutions to* (4.12) *and* (4.13) *are given by the following tables:*

Table 4.1: Primal and dual optimal sequences for (4.12) and (4.13), respectively.

Stage	v_t^*	q_t^*	g_t^*	Stage	μ_t^*	γ_t^*	u_t^*
$1 \leq t \leq i-1$	$v_0 - \theta_t$	d	0	$1 \leq t \leq i$	$-\alpha^{i-1}c$	$\alpha^{i-1}c$	0
$t = i$	0	$v_0 + a - \theta_{i-1}$	$\theta_i - v_0$	$t \geq i+1$	$-\alpha^{t-1}c$	$\alpha^{t-1}c$	0
$t \geq i+1$	0	a	$d - a$				

Proof. Suppose that $v_0 > d+\overline{v}-a$. From the state transition equation $v_t - v_{t-1} + q_t = a$, we have that $v_0 = v_1 + q - a \leq \overline{v} + d - a$, which is a contradiction. Hence, the primal problem is infeasible for any initial state v_0 greater than $d + \overline{v} - a$. For the dual problem (4.13), consider the following dual feasible sequence:

Stage	μ_t^k	γ_t^k	u_t^k
$t = 1$	k	$-k$	$-k$
$t > 1$	0	0	0

Therefore, the dual objective value is $k(v_0 + a - d - \overline{v})$, which diverges to $+\infty$ as k tends to $+\infty$. Thus, the dual problem (4.13) is unbounded.

Suppose that $v_0 \leq d + \overline{v} - a$. It is straightforward to check that that the sequences defined by table 4.1 are primal and dual feasible, respectively. We just need to show that the corresponding objective values are equal since weak duality holds by Lemma 4.4.1. Indeed, we have the following expression for the primal objective value:

$$
\begin{aligned}
\sum_{t=1}^{\infty} \alpha^{t-1} c \cdot g_t &= \alpha^{i-1} c \cdot (\theta_i - v_0) + \sum_{t=i+1}^{\infty} \alpha^{t-1} c \cdot (d-a) \\
&= \alpha^{i-1} c \cdot (\theta_i - v_0) + \frac{\alpha^i c}{(1-\alpha)}(d-a) \\
&= \alpha^{i-1} c \cdot \left[(i-1)(d-a) - v_0 + \frac{d-a}{(1-\alpha)} \right] \\
&= \alpha^{i-1} c \cdot \left[\frac{d-a}{(1-\alpha)} - (v_0 - \theta_{i-1}) \right]. \qquad (4.15)
\end{aligned}
$$

Similarly, we obtain the same expression for the dual objective value:

$$
\begin{aligned}
v_0 \mu_1 + \sum_{t=1}^{\infty} \left[d\gamma_t + a\mu_t + \overline{v} u_t \right] &= -v_0 \alpha^{i-1} c + \sum_{t=1}^{i} \alpha^{i-1} c[d-a] + \sum_{t=i+1}^{\infty} \alpha^{t-1} c[d-a] \\
&= \alpha^{i-1} c[-v_0 + i(d-a)] + \frac{\alpha^i c}{1-\alpha}[d-a] \\
&= \alpha^{i-1} c \cdot \left[-v_0 + i(d-a) + \frac{\alpha}{(1-\alpha)}(d-a) \right] \\
&= \alpha^{i-1} c \cdot \left[\frac{d-a}{(1-\alpha)} - (v_0 - \theta_{i-1}) \right].
\end{aligned}
$$

□

Corollary 4.5.1. *The value function V from* (4.12) *is an extended real-valued piecewise linear convex function that can be represented in the following form:*

$$
V(v_0) = \max_{i \geq 1} \left\{ \alpha^{i-1} c \cdot \left[\frac{d-a}{(1-\alpha)} - (v_0 - \theta_{i-1}) \right] \right\} + \mathbb{I}_{[0, d+\overline{v}-a]}(v_0), \qquad (4.16)
$$

where $\theta_i = i(d-a)$. In particular, if $\overline{v}$ is $+\infty$, then the value function V has an infinite *number of linear pieces.*

Proof. The value function V is an extended real-valued convex function since the dual problem (4.13) is the maximum of affine functions over v_0. Consequently, it follows from (4.15) the form for each linear piece for (4.16). □

Consider a primal infinite-dimensional stationary linear program with non-negative decisions x_t and z_t at each stage $t \geq 1$, a state transition equation $x_t + z_t - 2x_{t-1} = 0$, an unit cost of 0 for x_t and 1 for z_t with discount factor α equal to $1/2$:

$$\begin{aligned} V_C(x_0) := \inf \quad & \sum_{t=1}^{\infty} (1/2)^{t-1} z_t && (4.17a)\\ \text{s.t.} \quad & x_t + z_t - 2x_{t-1} = 0, \quad t \geq 1, && (4.17b)\\ & x_t, z_t \geq 0, \qquad\qquad\quad t \geq 1. && (4.17c) \end{aligned}$$

Note, we do not include, in (4.17), the $\ell_\infty(\mathbb{R})$ constraint for the sequence of decision variables $\boldsymbol{x}$ and $\boldsymbol{z}$. If we take the dual of (4.17) using the same duality rules for finite-dimensional linear programs we get the following problem:

$$\begin{aligned} W_C(x_0) := \sup \quad & 2x_0\mu_1 && (4.18a)\\ \text{s.t.} \quad & \mu_t - 2\mu_{t+1} \leq 0, \quad t \geq 1, && (4.18b)\\ & \mu_t \leq (1/2)^{t-1}, \quad\ \ t \geq 1, && (4.18c)\\ & \mu_t \in \mathbb{R}, \qquad\qquad\ t \geq 1. && (4.18d) \end{aligned}$$

Again, we do not have in (4.18) the $\ell_1(\mathbb{R})$ constraint for the sequence of dual variables $\boldsymbol{\mu}$. The proposition below presents the explicit form of the value functions V_C and W_C.

Proposition 4.5.2. *The value functions V_C and W_C have the following expressions:*

$$V_C(x_0) = \begin{cases} 0, & \textit{if } x_0 \geq 0,\\ +\infty, & \textit{otherwise,} \end{cases} \qquad W_C(x_0) = \begin{cases} 2x_0, & \textit{if } x_0 \geq 0,\\ +\infty, & \textit{otherwise.} \end{cases} \tag{4.19}$$

Therefore, problems (4.17) *and* (4.18) *do not satisfy weak duality.*

Proof. We first compute the value function V_C of (4.17). If x_0 is non-negative, the primal sequence defined as $(x_t, z_t) = 2^t(x_0, 0)$ for all $t \geq 1$ is feasible to (4.17) and attains the objective lower bound 0. Hence, $(\boldsymbol{x}, \boldsymbol{z})$ is an optimal primal sequence to (4.17) and $V_C(x_0)$ equals 0. If x_0 is negative, the feasible set of (4.17) is empty and thus $V_C(x_0)$ is equal to $+\infty$.

We now compute the value function W_C of (4.18). If x_0 is non-negative, the dual sequence defined as $\mu_t = (1/2)^{t-1}$ for all $t \geq 1$ is a feasible to (4.18) and μ_1 achieves its upper bound of 1. Hence, $\boldsymbol{\mu}$ is a dual optimal sequence and $W_C(x_0)$ equals $2x_0$. If x_0 is negative, then for each $k \geq 1$ and $t \geq 1$, the dual sequence defined as $\mu_t^k = -k(1/2)^{t-1}$ is feasible to (4.18) and it has dual objective value of $-2kx_0$. Thus, problem (4.18) is unbounded and $W_C(x_0)$ is equal to $+\infty$. □

Surprisingly, if we consider the $\ell_\infty(\mathbb{R})$ constraint to the primal problem (4.17) and the $\ell_1(\mathbb{R})$ to the dual problem (4.18) we recover strong duality as illustrated in the next Proposition.

Proposition 4.5.3. *Let V be the value function defined by* (4.17) *with the $\ell_\infty(\mathbb{R})$ constraint on the primal sequences $\boldsymbol{x}$ and $\boldsymbol{z}$ and let W be the value function defined by* (4.18) *with the $\ell_1(\mathbb{R})$ constraint on the dual sequence $\boldsymbol{\mu}$. Then, the value functions V and W have the following form:*

$$V(x_0) = W(x_0) = \begin{cases} 2x_0, & \text{if } x_0 \geq 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.20}$$

In particular, the primal and dual problems (4.17) *and* (4.18) *with the $\ell_\infty(\mathbb{R})$ and $\ell_1(\mathbb{R})$ constraints, respectively, satisfy strong duality.*

Proof. The argument to prove the expression for $W(x_0)$ is analogous to the argument in the proof of Proposition 4.5.2 for $W_C(x_0)$ since all the dual sequences $\boldsymbol{\mu}$ used in the previous proof are $\ell_1(\mathbb{R})$ bounded.

From Lemma 4.4.1, we know that weak duality holds for $V(x_0)$ and $W(x_0)$. Thus, the inequality $W(x_0) \leq V(x_0)$ holds for all $x_0 \in \mathbb{R}^n$. If x_0 is non-negative, then the sequence defined as

$$(x_t, z_t) := \begin{cases} (0, 2x_0), & \text{if } t = 1, \\ (0, 0), & \text{if } t \geq 2, \end{cases}$$

for all $t \geq 1$ is feasible to (4.17), $(\boldsymbol{x}, \boldsymbol{z})$ belongs to $\ell_\infty(\mathbb{R}) \times \ell_\infty(\mathbb{R})$, and the associated objective value is $2x_0$. Since $W(x_0)$ equals $2x_0$, we conclude that the primal sequence $(\boldsymbol{x}, \boldsymbol{z})$ is optimal and $V(x_0) = 2x_0$. If x_0 is negative, then the feasible set of (4.17) is empty and thus $V(x_0)$ is equal to $+\infty$. □

4.5.3 Lot-Sizing as a generalization of the Hydro-Thermal planning problem

In this section, we comment about the similarity between the Stationary Hydro-Thermal Power Planning problem from the previous section and the Stationary Single-Item Lot-Sizing problem described in [56]. In fact, the Hydro-Thermal problem (4.12) can be framed as a Single-Item Lot-Sizing. A major difference is that the latter problem has a nonlinear objective function, so our duality framework does not hold for it.

Let z_t and s_t be the production and stock at time t, respectively. The objective function is composed by the unit cost $c \geq 0$ to produce an item, the setup cost $f \geq 0$ for production, and the holding cost $h \geq 0$ for the stock. The constraint parameters are the single-item demand $d \geq 0$ in each time period, the maximum production capacity $\overline{z}$, the maximum stock capacity $\overline{s}$, and the initial stock $s \geq 0$. The Stationary Single-Item Lot-Sizing problem is defined as the following *nonlinear* problem:

$$C(s_0) = \inf_{\boldsymbol{z}, \boldsymbol{s}} \sum_{t=1}^{\infty} \alpha^{t-1} \big(f H(z_t) + c z_t + h s_t \big) \tag{4.21a}$$

$$\text{s.t.} \quad z_t + s_{t-1} - s_t = d, \qquad \forall t \geq 1, \tag{4.21b}$$

$$z_t \leq \overline{z}, \quad s_t \leq \overline{s}, \qquad \forall t \geq 1, \tag{4.21c}$$

$$z_t, s_t \geq 0, \qquad \forall t \geq 1, \tag{4.21d}$$

(4.21e)

where $\alpha \in (0, 1)$ is the discount factor, and $H(z)$ is the Heaviside function, that is,

$$H(z) := \begin{cases} 1, & \text{if } z > 0, \\ 0, & \text{if } z \leq 0. \end{cases} \tag{4.22}$$

If we replace the hydro generation variable q_t by the right-hand side of the identity $q_t = d - g_t$ then we frame the Stationary Hydro-Thermal Power Planning Problem (4.12) as a single-item lot-sizing problem (4.21):

$$V(v_0) = \inf \quad \sum_{t=1}^{\infty} \alpha^{t-1} \cdot c g_t \tag{4.23a}$$

$$\text{s.t.} \quad g_t + v_{t-1} - v_t = d - a, \quad t \geq 1, \tag{4.23b}$$

$$g_t \leq d, \quad v_t \leq \overline{v}, \quad t \geq 1, \tag{4.23c}$$

$$g_t, v_t \geq 0, \quad t \geq 1, \tag{4.23d}$$

Note that the setup cost f is 0 for (4.23).

Single-Item Uncapacitated Lot-Sizing problem

[56] analyzes the Single-Item Uncapacitated, i.e., the Lot-Sizing (4.21) with production $\overline{z}$ the stock $\overline{s}$ maximum capacities equal to $+\infty$. They approach this problem using only primal techniques as described by the following theorem:

Theorem 4.5.1. *Suppose $d > 0$, and let $t^* = \lfloor \frac{s}{d} \rfloor + 1$. Any optimal solution to* (4.21) *with $\overline{z} = \overline{s} = +\infty$ satisfies the following statements:*

1. $z_t = 0$, *for all $t < t^*$.*

2. $z_{t^*} > 0$, *and $s_{t^*-1} + z_{t^*} = k_{t^*} d$, for some $k_{t^*} \in \mathbb{N}$.*

3. $s_{t-1} z_t = 0$, *for all $t > t^*$, and if $z_t > 0$, then $z_t = k_t d$, for some $k_t \in \mathbb{N}$.*

Proof. See [56]. □

Theorem 4.5.1 states that any optimal solution to the Single-Item Uncapacitated Lot-Sizing problem must use as possible the inventory s_t to meet the demand d without any production, and then produce enough items so that the next inventory s_{t^*} is an integer multiple of the demand.

One issue with the proof of Theorem 4.5.1 is whether or not there exists any optimal solution. The second issue is how Theorem 4.5.1 is used to establish the expression for the value function $C(s_0)$ using dynamic programming. Again, the existence of an optimal solution is necessary to establish a dynamic programming expression.

For simplicity, let us focus on the case when the initial inventory s_0 is 0. If the optimal solution exists then it should be of the form

$$z_t^n = \begin{cases} nd, & \text{if } (t-1)\%n = 0, \\ 0, & \text{if } (t-1)\%n > 0, \end{cases} \quad \text{and} \quad s_t^n = (n-t)d, \tag{4.24}$$

for some $n \in \mathbb{N}$. We call n the *replenishment interval*. This implies that we can restrict the Single-Item Uncapacitated Lot-Sizing (4.21) to feasible solutions (4.24) parameterized by n: $C(0) = \inf_{n\in\mathbb{N}} \sum_{t=1}^{\infty} \alpha^{t-1}\big(f\delta(z_t^n) + cz_t^n + hs_t^n\big)$. By expanding this expression, we obtain the following formula:

$$\begin{aligned} C(0) &= \inf_{n\in\mathbb{N}} \left\{ \frac{1}{1-\alpha^n} \left(f + ncd + hd \sum_{l=1}^{n-1} \alpha^{l-1}(n-l) \right) \right\} \\ &= \inf_{n\in\mathbb{N}} \left\{ \frac{n\left(cd + \frac{hd}{(1-h)}\right) + f}{1-\alpha^n} - \frac{hd}{(1-\alpha)^2}, \right\}, \end{aligned} \tag{4.25}$$

where the second equality comes from the identity $\sum_{l=1}^{k} \alpha^{l-1}(k-l) = \frac{k(1-\alpha)-(1-\alpha^k)}{(1-\alpha)^2}$.

Note that the infimum of (4.25) is not attainable if $c = h = 0$. Indeed,

$$C(0) = \inf_{n\in\mathbb{N}} \frac{f}{1-\alpha^n} = \lim_{n\to\infty} \frac{f}{1-\alpha^n} = f.$$

A more intuitive way to understand this fact is to analyze the feasible solutions. For every feasible solution the production z_1 at the first time period must be positive since the initial inventory is 0, and there is another positive production z_t at some future time period $t > 1$. Thus, the objective cost of any feasible solution must be greater than f. Finally, each feasible solution $(\boldsymbol{z}^n, \boldsymbol{s}^n)$ defined in (4.24) has objective cost $\frac{f}{1-\alpha^n}$, which converges to f as n tends to ∞. Hence, if $c = h = 0$ then the infimum of (4.25) is not attainable.

[56] also observes that the right-hand side expression (4.25) as a continuous function of n is Strongly Convex if the production cost c or the holding cost h is positive. Then, one can find the optimal replenishment interval n by finding the optimal continuous solution n^* to (4.25) and by taking the value among the round-up $\lceil n^* \rceil$ and round-down $\lfloor n^* \rfloor$ solutions that has the minimum objective cost.

A conjecture for the Periodic Uncapacitated Single-Item Lot-Sizing problem

Inspired by dynamic programming idea of [56], it might be possible to extend their solution method to a larger class of infinite-horizon programs such as the Hydro-Thermal Power Planning problem or Single-Item Lot-Sizing problem with periodic coefficients modulo T, where $T \in \mathbb{N}$ is a fixed number. However, the generalization of Theorem 4.5.1 may not be straightforward for the periodic case since its proof directly uses the stationarity property of the coefficients.

Consider the following Single-Item Lot-Sizing problem with periodic coefficients modulo T:

$$C^P(s_0) = \inf_{\boldsymbol{z},\boldsymbol{s}} \quad \sum_{t=1}^{\infty} \alpha^{t-1}\big(f_t H(z_t) + c_t z_t + h_t s_t\big) \tag{4.26a}$$

$$\text{s.t.} \quad z_t + s_{t-1} - s_t = d_t, \qquad \forall t \geq 1, \tag{4.26b}$$

$$z_t \leq \overline{z}_t, \quad s_t \leq \overline{s}_t, \qquad \forall t \geq 1, \tag{4.26c}$$

$$z_t, s_t \geq 0, \qquad \forall t \geq 1, \tag{4.26d}$$

(4.26e)

where all the coefficients are periodic modulo T, that is,

$$f_t = f_{t+T}, \quad c_t = c_{t+T}, \quad h_t = h_{t+T}, \quad d_t = d_{t+T}, \quad \overline{z}_t = \overline{z}_{t+T}, \quad \text{and} \quad \overline{s}_t = \overline{s}_{t+T}, \tag{4.27}$$

for all $t \geq 1$. For simplicity, we only analyze for the case where the initial state s_0 is equal to 0. Below we introduce an auxiliary problem whose optimal solutions generalize the replenishment interval solutions (4.24):

$$\begin{aligned} C_n^P(0) := \min_{z,s} \quad & \sum_{t=1}^{nT} \alpha^{t-1}\big(f_t H(z_t) + c_t z_t + h_t s_t\big) && \text{(4.28a)} \\ \text{s.t.} \quad & z_t + s_{t-1} - s_t = d_t, \qquad \forall 1 \leq t \leq n, && \text{(4.28b)} \\ & s_0 = 0, \quad s_n = 0, && \text{(4.28c)} \\ & z_t \leq \overline{z}_t, \quad s_t \leq \overline{s}_t, \qquad \forall 1 \leq t \leq n, && \text{(4.28d)} \\ & z_t, s_t \geq 0, \qquad \forall 1 \leq t \leq n, && \text{(4.28e)} \end{aligned}$$

where n here is the “replenishment period”. We claim that when the infimum of (4.26) is attainable, there exists $n \in \mathbb{N}$ such that

$$C^P(0) = \frac{C_n^P(0)}{1 - \alpha^{nT}}. \tag{4.29}$$

We show some numerical examples that illustrates our ideas for the stationary case, that is, when $T = 1$. Let $\alpha = 0.95$, $d = 40$, $c = 3$, $h = 0.05$. We perform a sensitivity analysis for the setup cost f. The higher the setup cost the larger tends to be the production, and the longer tends to be the replenishment interval n. To evaluate our ideas, we solve the finite-horizon version (4.26) and call its optimal value *longHorizon*. The longHorizon problem is the same as (4.28), where n equals 300. We also find the optimal replenishment interval n as described before with the round-up and round-down solutions of the optimal continuous solution n^* of (4.25). This leads to a feasible solution whose objective cost we call *lotPolicy*. Using the same replenishment interval n, we solve our heuristic (4.28) and compute the right-hand side of (4.29). We call such value *periodicPolicy*. We have a variable *withinError* that represents with True if all the costs of that particular instance are within a relative error of 10^{-4} of the lotPolicy cost, or False otherwise. Below, we present Table 4.2 with our sensitivity analysis. We performed a similar numerical experiment for

Table 4.2: Costs of each approach to solve the Uncapacitated Single-Item Lot-Sizing problem.

setupCost	longHorizon	lotPolicy	periodicPolicy	withinError	replenishment (n)
10	2584.62	2584.62	2584.62	True	2
50	2916.08	2916.04	2916.04	True	3
100	3178.46	3178.45	3178.45	True	5
200	3575.86	3575.75	3575.75	True	7
400	4176.52	4176.33	4176.33	True	9
500	4433.48	4433.48	4433.48	True	10
1000	5524.13	5524.11	5524.11	True	14
5000	11632.38	11632.37	11632.37	True	27

the periodic case, that is, when $T > 1$, and we observe the same agreement between the longHorizon optimal values and the periodicPolicy rescaled costs. This supports our claim that the approach (4.28) generalizes the explicit policy (4.24) for the periodic case.

CHAPTER 5

BASIC FEASIBLE SOLUTION FOR ROW-FINITE LINEAR SYSTEMS

This chapter investigates an algebraic method to characterize extreme points for convex sets defined by row-finite linear systems. Our initial motivation is whether or not the primal and dual optimal solutions of the stationary hydro-thermal planning problem are extreme points.

5.1 Related work

Extreme points have been studied in the literature of infinite dimensional linear programs for a long time but with most applications limited to network flow and non-stationary Markov decision process. In this section, we describe some papers that are directly related to extreme points for infinite dimensional linear programs.

Sufficient conditions for the existence of extreme optimal solutions for infinite horizon problems with Leontief constraints were studied in [58]. [59] develops the concept of right analytic extreme points for continuous-time linear programming, which is a type of full-rank sufficient condition. The work of [60] investigates the convergence in product topology sense of finite-dimensional projections of extreme points. They also extend the notion of total unimodularity to infinite systems of linear equalities and nonnegative variables. [11] establishes necessary and sufficient conditions for a network flow to be extreme on a graph with countably infinity nodes and finite degrees on each node. Inspired by the notion of basic feasible solutions, [10] establishes sufficient conditions for a solution to be an extreme point. They apply their concept to infinite network flow problems and non-stationary Markov decision processes. [61] provides a characterization of extreme point solution of non-stationary Markov decision processes with discounted cost criterion and finite state space.

Along with the investigations of properties of extreme points, there were several developments of simplex-type algorithms for infinite dimensional linear programs. A simplex method extension for semi-infinite linear programs was developed by [62], but such a method has some unresolved numerical issues. An infinite network simplex method was developed by [63] based on a notion of duality for network flow problems. Their algorithm takes a finite amount of time for each pivot step. The work of [64] proposes a simplex-type algorithm for a Countably Infinite Linear Programs class. It guarantees convergence for the class of Nonstationary Infinite-Horizon Markov Decision problems. [65] provide a different simplex method for a structured class of uncapacitated countably infinite network flow problems. It uses a primal approach based on the nonnegativity of reduced costs and convergence of spanning trees.

5.2 Contributions

1. **Asymptotically Compatible vectors and basic feasible solutions:** The geometric definition of an extreme point may not be a convenient method to certify whether or not a given point of a convex set is extreme. Given arbitrary linear constraints, we extend the definition of a basic feasible solution using Asymptotically Compatible (AC) vectors. We show that a point is extreme if and only if the *AC solution* to the linear equality system induced by the set of active constraints is the only the trivial solution. This method directly proves that the primal and dual optimal solutions of the stationary hydro-thermal power generation planning problem are extreme.

2. **Row-finite linear systems:** A row-finite linear system over the sequence space of real numbers is a countable set of linear constraints induced by coefficients with a finite number of non-zero elements. The Gauss-Jordan elimination method of [13] can parameterize all the solutions of a row-finite *equality* system. Such a method may certify the existence of a unique (or multiple) AC solution for the set of active constraints, which implies that the corresponding feasible solution is (not) extreme.

This idea is a direct parallel with the Gaussian elimination method for equality linear systems of finite dimension, but it has the rightmost pivoting as an important distinction.

3. **Application to extreme flows over countably infinite graphs:** We illustrate the use of our extreme point result for an alternative proof of the extreme flow characterization in countably infinite graphs of finite node degrees. The original result is from [11]. It provides another condition on the residual graph together with not having a cycle form the necessary and sufficient conditions for a network flow to be extreme.

5.3 An extension of basic feasible solution for arbitrary linear systems.

In this section, we introduce the notion of an *asymptotically compatible* (AC) solution and extend the definition of a *basic solution* for arbitrary systems of linear constraints. Finally, we conclude this section with the proof that an extreme point is equivalent to a basic feasible solution.

Let X be a vector space over $\mathbb{R}$ and consider a convex set $P \subseteq X$ represented by a family of linear constraints:

$$P = \left\{ x \in X \;\middle|\; \begin{array}{ll} F_i(x) = b_i, & \forall i \in I, \\ G_j(x) \leq h_j, & \forall j \in J. \end{array} \right\}, \tag{5.1}$$

where F_i and G_j are linear real-valued functions on X, I and J are arbitrary index sets, and b_i and h_j are real numbers. We say that $x \in P$ is an *extreme point* if there is no line segment in P that contains x as mid-point, or, in other words, for every $w, y \in P$ such that $\frac{1}{2}(w + y) = x$ it implies that $y = w = x$.

For each inequality constraint $j \in J$, denote by $r_j(x)$ the slack function defined by the difference $h_j - G_j(x)$. Let N_x^+ and N_x^0 be the index sets of non-binding and binding inequality constraints at $x \in P$, respectively, that is, $N_x^+ := \{j \in J \mid r_j(x) > 0\}$ and $N_x^0 := \{j \in J \mid r_j(x) = 0\}$. We say that a vector $d \in X$ is *asymptotically compatible* with $x \in X$ (AC-x) if

$$\sup_{j \in N_x^+} \frac{|G_j(d)|}{r_j(x)} < +\infty. \tag{5.2}$$

In particular, 0 and x are AC-x vectors. Note that the set of AC-x vectors forms a linear subspace of X. Recall that a vector $d \in X$ is a feasible direction at $x \in P$ if there exists a positive scalar $\alpha > 0$ such that $x + \alpha d$ belongs to P.

Lemma 5.3.1 (Characterization of feasible directions)**.** *Consider the following system on* $(x, d) \in P \times X$*:*

$$F_i(d) = 0, \quad \forall i \in I, \tag{5.3a}$$

$$G_j(d) \leq 0, \quad \forall j \in N_x^0, \tag{5.3b}$$

The direction $d \in X$ *is feasible at* $x \in P$ *if, and only if, the tuple* (x, d) *satisfies the system* (5.3) *and the supremum* $c := \sup_{j \in N_x^+} \frac{G_j(d)}{r_j(x)}$ *is not* $+\infty$*. In particular, the set* $\mathcal{I}$ *of feasible steps along* d *is*

$$\mathcal{I} = \begin{cases} [0, 1/c], & \textit{if } c > 0, \\ [0, \infty), & \textit{if } c \leq 0. \end{cases}$$

Proof. Suppose that $d \in X$ is a feasible direction at $x \in P$, i.e., there exists $\alpha > 0$ such that $x + \alpha u \in P$. Then, the pair (x, d) satisfies (5.3). We now check that c is not $+\infty$. Indeed, $G_j(x + \alpha d) \leq h_j$ for all $j \in J$, which implies that

$$\frac{G_j(d)}{r_j(x)} \leq \frac{1}{\alpha}, \quad \forall j \in N_x^+.$$

If N_x^+ is empty then c is $-\infty$, by convention. Hence, c is not $+\infty$.

Conversely, suppose that (x, d) satisfies (5.3) and $c < +\infty$. If c is $-\infty$, which is equivalent to say that N_x^+ is empty, then $G_j(d)$ is non-positive for all $j \in J$ because N_x^0 equals the whole index set J. If c is a non-positive real number then $G_j(d)$ is also non-positive for all $j \in J$ because c is the supremum of the ratio $G_j(d)/r_j(x)$ over $j \in N_x^+$ and r_j is non-negative at x. In both cases, d is a feasible direction at x and the set of feasible steps $\mathcal{I}$ along d is $[0, \infty)$.

If c is a positive real number then there exists a function $G_j(d)$ which is positive, so we can represent c as the supremum $c = \sup_{j \in J:\ G_j(d) > 0} \frac{G_j(d)}{r_j(x)}$.

Let $\overline{\alpha}$ be the constant $1/c$ and note that:

$$\overline{\alpha} = \frac{1}{\displaystyle\sup_{j \in J: G_j(d) > 0} \frac{G_j(d)}{r_j(x)}} = \inf_{j \in J: G_j(d) > 0} \frac{r_j(x)}{G_j(d)}. \tag{5.4}$$

We prove that $G_j(x + \overline{\alpha} d)$ is non-positive for all $j \in J$ and conclude that d is a feasible direction. Indeed, if $G_j(d)$ is non-positive then $G_j(x + \overline{\alpha} d) \leq G_j(x) \leq h_j$. If $G_j(d)$ is positive then we have that

$$\begin{aligned} G_j(x + \overline{\alpha} d) &= G_j(x) + \left(\inf_{k \in J: G_k(d) > 0} \frac{r_k(x)}{G_k(d)} \right) \cdot G_j(d) \\ &\leq G_j(x) + \frac{r_j(x)}{G_j(d)} G_j(d) = h_j. \end{aligned}$$

Hence, d is a feasible direction.

We now prove that for any α greater than $\overline{\alpha}$ the point $x + \alpha d$ does not belong to P. Indeed, from the infinmum property, there exists $j \in J$ such that $G_j(d)$ is positive and $\overline{\alpha} \leq \frac{r_j(x)}{G_j(d)} < \alpha$. This implies that $x + \alpha d$ violates the j-th inequality constraint:

$$h_j = G_j(x) + \frac{r_j(x)}{G_j(d)} G_j(d) < G_j(x) + \alpha G_j(d) = G_j(x + \alpha d).$$

Hence, $x + \alpha d \notin P$ and $\mathcal{I} = [0, 1/c]$. □

Using Lemma 5.3.1, we can extend the following characterization of extreme points.

Theorem 5.3.1 (Extreme Point Characterization Theorem)**.** *Let $P \subseteq X$ be a convex set as in* (5.1) *and let $x \in P$. Then, the following are equivalent:*

1. *x is an extreme point.*

2. *The unique AC-x solution to the homogeneous equality linear system*

$$F_i(d) = 0, \quad \forall i \in I, \tag{5.5a}$$

$$G_j(d) = 0, \quad \forall j \in N_x^0, \tag{5.5b}$$

   is the zero vector.

3. *The unique AC-x solution to the linear equality system*

$$F_i(x') = b_i, \quad \forall i \in I, \tag{5.6a}$$

$$G_j(x') = h_j, \quad \forall j \in N_x^0, \tag{5.6b}$$

   is the solution x.

Proof. $(1) \implies (2)$: Suppose that x is an extreme point of P and let $d \in X$ be an AC-x solution to the homogeneous system (5.5). From Lemma 5.3.1, we have that d and $-d$ are both feasible directions. In particular, there exists a positive scalar α such that the vectors $x + \alpha d$ and $x - \alpha d$ belong to P, which implies that

$$x = \frac{1}{2}(x + \alpha d) + \frac{1}{2}(x - \alpha d).$$

Since x is an extreme point, we have that $x + \alpha d = x - \alpha d$. Thus, d is the 0 vector.
$(2) \implies (3)$: Suppose the only AC-x solution to the homogeneous linear system (5.5) is the zero vector. Let y and w be two AC-x solutions to the inhomogeneous system (5.6). This implies that the difference of vectors $y - w$ is an AC-x solution to the homogeneous linear system (5.5), so y equals w.

$(3) \implies (1)$: Suppose the only AC-x solution to the inhomogeneous linear system (5.6) is x. Let $y, w \in P$ be such that x is the midpoint vector in the line segment between y and w, that is, $\frac{y+w}{2} = x$. Let d be the direction from w to y, i.e., $d = (y-w)/2$ and note that d and $-d$ are both feasible directions at x. From Lemma 5.3.1, the direction d is an AC-x solution to the homogeneous linear system (5.5) and because y equals $x + d$ and w equals $x - d$ we have that the vectors y and w are AC-x solutions to the inhomogeneous linear system (5.6). Hence, the vectors y and w are equal to x. □

After Theorem 5.3.1, it makes sense to extend the notion of a basic solution to arbitrary linear systems. We say that a vector $x \in X$ is a *basic solution* to P if x is the unique AC-x solution to the inhomogeneous system (5.6) induced by the binding constraints at x. Note that x may not be a point in P. We say that x is a basic *feasible* solution to P if x is a basic solution and x belongs to P.

5.3.1 Example: the primal and dual optimal solutions of the stationary hydro-thermal power planning problem

The extreme point characterization from Theorem 5.3.1 can be used to prove that the primal and dual optimal solutions from Table 4.1 are extreme points.

Corollary 5.3.1 (Hydro-Thermal optimal solutions)**.** *The primal and dual optimal solutions from Table 4.1 are extreme points.*

Proof. The primal and dual equality linear systems induced by the binding constraints at the solutions of Table 4.1 are the following:

$$\begin{cases} v_t - v_{t-1} + q_t = a, & t \geq 1, \\ g_t + q_t = d, & t \geq 1, \\ g_t = 0, & 1 \leq t \leq i-1, \\ v_t = 0, & t \geq i, \end{cases} \quad \text{and} \quad \begin{cases} \mu_t - \mu_{t+1} + u_t = 0, & 1 \leq t \leq i-1, \\ \mu_t + \gamma_t = 0, & t \geq 1, \\ \gamma_t = \alpha^{t-1} c, & t \geq i, \\ u_t = 0, & t \geq 1, \end{cases} \tag{5.7}$$

where i is the smallest index $k \geq 0$ such that $v_0 \leq k(d-a)$. It is straightforward to see that both linear systems in (5.7) have *exactly* one solution which corresponds to the primal and dual optimal solutions from Table 4.1, respectively. It follows from Theorem 5.3.1 that those primal and dual optimal solutions are extreme points. □

For more general problems, it may not straightforward to identify the AC solutions of an equality linear system. In the next section, we provide a general technique to parameterize all the solutions of a particular equality linear system called

5.4 Row-finite linear systems

This section investigates a more systematic approach to determine if a vector is a basic feasible solution based on the constraint coefficients and right-hand side vector. In particular, we restrict our scope to systems of linear constraints defined by row-finite matrices following [13]; each constraint has only a finite number of non-zero coefficients. By finding all the solutions to a row-finite linear system, we may be able to check whether there is a non-trivial AC-x solution to it or not.

Let $c_{00}(\mathbb{R})$ be the set of vectors over $\mathbb{R}^\omega$ that have a finite number of non-zero entries. Let $\mathrm{RFM}(\mathbb{R})$ be the space of infinite matrices where each matrix $\mathcal{A} \in \mathrm{RFM}(\mathbb{R})$ has a countable number of rows and columns, and each row of $\mathcal{A}$ is a vector in $c_{00}(\mathbb{R})$. Consider the algebra over $\mathrm{RFM}(\mathbb{R})$ formed by the usual addition, scalar, and matrix multiplication. We define the *length* function $l : c_{00}(\mathbb{R}) \to \mathbb{Z}_+$ as the position of the last nonzero coordinate of a given vector $u \in c_{00}(\mathbb{R})$. If u is the zero vector we define $l(u)$ as 0.

We denote by calligraphic letters such as $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ the row-finite matrices in $\mathrm{RFM}(\mathbb{R})$, while the regular capital letters such as A, B, and C are reserved for finite dimensional matrices in $\mathbb{R}^{m\times n}$. Let a_i denote the i-th row of $\mathcal{A}$, and let a_{ij} denote the ij-entry of $\mathcal{A}$.

5.4.1 Solution parametrization and the Hermitian Normal Form

In this section, we introduce an important class of row-finite matrix to parameterize the solutions of equality linear systems called Hermitian Normal Form.

We say that a row-finite matrix $\mathcal{H} \in \mathrm{RFM}(\mathbb{R})$ is in the Hermitian Normal Form (HNF) if the following conditions hold:

1. The length of the nonzero rows of $\mathcal{H}$ is in increasing order, that is, $l(h_i) < l(h_j)$, if $i < j$ and h_i, h_j are both non-zero rows.
2. The rightmost coefficient of any nonzero row h_i of $\mathcal{H}$ is 1, that is, $h_{i,l(h_i)} = 1$.
3. The coefficients above and below the rightmost coefficient of a nonzero row h_i are 0, that is, $h_{k,l(h_i)} = 0$, for every $k \neq i$.

It is simple to obtain all the solutions of a linear system $\mathcal{H}x = b$ if $\mathcal{H}$ is in HNF. Indeed, we can partition the set of natural numbers into those such that the length of the i-th is different than zero and those that are zero, respectively. Then, we describe the linear system $\mathcal{H}x = b$ equivalently as

$$
\begin{aligned}
x_{l(h_i)} &= b_i - \sum_{j=1}^{l(h_i)-1} h_{ij} x_j, && \forall i \in \mathbb{N};\ l(h_i) \neq 0, \\
0 &= b_i, && \forall i \in \mathbb{N};\ l(h_i) = 0,
\end{aligned}
$$

where h_{ij} is the ij-th entry of $\mathcal{H}$. So, the linear system $\mathcal{H}x = b$ has a solution if and only if b_i is 0 for every row h_i of zero length. The solutions to $\mathcal{H}x = b$ can be parameterized by the coordinates $x_j \in \mathbb{R}$ such that $j \neq l(h_i)$, for all $i \in \mathbb{N}$.

An example of extreme point over $\mathbb{R}^\omega$.

The following example was inspired by [66]. Consider the convex set $P \subseteq \mathbb{R}^\omega$ defined by the following system of row-finite linear constraints:

$$
\begin{array}{lllllll}
x_1 + x_2 & & & & & & = \ 2, \\
x_1 & - x_3 & & & & & = 1/4, \\
& x_2 & - x_4 & & & & = 1/4, \\
& & x_3 & - \ x_5 & & & = 1/8, \\
& & & x_4 & - x_6 & & = 1/8, \\
& & & \ddots & & \ddots & \vdots \\
& & x_j \geq 0, & \forall j \in \omega. & & &
\end{array}
\tag{5.8}
$$

Let $\mathcal{A}$ be the row-finite constraint matrix and let b be the right-hand side vector defined by the equality system (5.8). Let $x^* \in \mathbb{R}^\omega$ be a vector defined as

$$
x_j^* = \begin{cases} (2^k + 1)/2^k, & \text{if } j = 2k - 1 \text{ for some } k \geq 1, \\ 1/2^k, & \text{if } j = 2k \text{ for some } k \geq 1. \end{cases}
$$

It is straightforward to check that x^* is a solution to (5.8), so it belongs to the convex set P. However, checking whether x^* is an extreme point might be challenging. A more convenient approach is to perform elementary row operations to transform the row-finite matrix $\mathcal{A}$ into its HNF, parameterize all the solutions to the linear system (5.8), and apply the Basic Feasible Solution characterization of Theorem 5.3.1.

Indeed, consider the homogeneous linear system obtained from (5.8). We cancel the non-rightmost coefficient of the i-th constraint using the rightmost coefficient of the $(i-2)$-th. Starting at i equal to 3 and proceeding successfully for all constraints $i \geq 3$, we obtain the following equivalent homogeneous linear system:

$$\begin{array}{lllllll} d_1 & + d_2 & & & & & = 0, \\ d_1 & & - d_3 & & & & = 0, \\ -d_1 & & & - d_4 & & & = 0 \\ d_1 & & & & - d_5 & & = 0, \\ -d_1 & & & & & - d_6 & = 0, \\ \vdots & & & & & \ddots & \vdots \end{array} \tag{5.9}$$

The formal justification of (5.9) requires a transfinite induction but the idea is quite intuitive. The row-finite matrix $\mathcal{H}$ defined by (5.9) only violates the condition 2 on the HNF definition. Indeed, the normalization of the right-most coefficient of each row is only to ensure *uniqueness* of the HNF decomposition up to permutations of the rows of zero length, see [67] and [13].

Thus, any homogeneous solution d to (5.9) is parameterized by $d_1 \in \mathbb{R}$ as $d_{2k} = -d_1$ and $d_{2k+1} = d_1$, for all $k \in \mathbb{N}$. In particular, we compute the AC-x^* condition for any homogeneous solution d:

$$\begin{aligned} \sup_{i\in\mathbb{N}} \frac{|d_i|}{x_i^*} &= \max\left\{\sup_{k\in\mathbb{N}} \frac{|d_1|}{(2^k+1)/2^k}\ ,\ \sup_{k\in\mathbb{N}} \frac{|d_1|}{1/2^k}\right\} = \max\left\{|d_1|\ ,\ \sup_{k\in\mathbb{N}} 2^k|d_1|\right\} \\ &= \begin{cases} 0, & \text{if } d_1 = 0, \\ +\infty, & \text{if } d_1 \neq 0. \end{cases} \end{aligned}$$

Because the only AC-x^* solution to the homogeneous linear system (5.9) is the zero vector, we conclude that x^* is an extreme point of P, by Theorem 5.3.1.

5.4.2 The Gauss-Jordan elimination method for row-finite matrices

In this section, we present the Gauss-Jordan elimination method to find the HNF decomposition of any row-finite matrix, as introduced by [13]. Any row-finite matrix $\mathcal{A} \in \mathrm{RFM}(\mathbb{R})$

can be decomposed as $\mathcal{H} = \mathcal{Q}\mathcal{A}$, where $\mathcal{H} \in \mathrm{RFM}(\mathbb{R})$ is in HNF and $\mathcal{Q} \in \mathrm{RFM}(\mathbb{R})$ is non-singular. We provide an alternative construction of $\mathcal{Q}$ based on a sequence of matrices with increasing order. Let $\mathrm{RFM}_k(\mathbb{R})$ be the space of row-finite matrices with k rows. Below we describe the Gauss-Jordan elimination method with rightmost pivoting rule:

1. Initialization: let $\mathcal{H}^{(0)}, Q^{(0)} := \emptyset$, and let $k = 1$.

2. For all $k \geq 1$ do:

   (a) *Gaussian step*. Consider the row-finite matrix $\left[\begin{smallmatrix} \mathcal{H}^{(k-1)} \\ a_k \end{smallmatrix}\right] \in \mathrm{RFM}_k(\mathbb{R})$. Using the rightmost nonzero entry of the first $k-1$ rows, apply elementary row operations to vanish the corresponding coordinates of a_k. This operation is equivalent to a left-multiplication by a non-singular lower triangular matrix $G^{(k)} \in \mathbb{R}^{k \times k}$.

   (b) *Normalization step*. After the Gaussian step, we have the row-finite matrix $\left[\begin{smallmatrix} \mathcal{H}^{(k-1)} \\ g_k \end{smallmatrix}\right]$. If g_k is a nonzero vector we normalize it by its rightmost nonzero coefficient. This operations is equivalent to a left-multiplication by a non-singular diagonal matrix $N^{(k)} \in \mathbb{R}^{k \times k}$.

   (c) *Jordan step*. After the normalization step, we obtain the row-finite matrix $\left[\begin{smallmatrix} \mathcal{H}^{(k-1)} \\ h_k \end{smallmatrix}\right]$, where the rightmost coefficient $h_{k,l(h_k)}$ is 1 if h_k is a nonzero vector. We clear the elements above the k-th row at the $l(h_k)$-th column. This operation is equivalent to a left-multiplication by a non-singular upper triangular matrix $J^{(k)} \in \mathbb{R}^{k \times k}$.

   (d) *Permutation step*. After the Jordan step, we obtain a row-finite matrix $\left[\begin{smallmatrix} \widetilde{\mathcal{H}}^{(k-1)} \\ h_k \end{smallmatrix}\right]$ that is almost in HNF. We order the nonzero rows to an increasing order of length and do not change the position of the zero rows. This operation is equivalent to a left-multiplication by a permutation matrix $P^{(k)} \in \mathbb{R}^{k \times k}$. The resulting row-finite matrix is denoted by $\mathcal{H}^{(k)}$ and it is HNF.

   (e) Update the matrix $Q^{(k)}$. Denote by E_k the elementary matrix obtained by the steps from (a)-(d), that is, $E_k := P^{(k)} J^{(k)} N^{(k)} G^{(k)}$. Define $Q^{(k)} \in \mathbb{R}^{k \times k}$ using the recursive formula below:

$$Q^{(k)} := E_k \begin{bmatrix} Q^{(k-1)} & \\ & 1 \end{bmatrix}. \tag{5.10}$$

The matrices $G^{(k)}$ and $J^{(k)}$ for the Gaussian and Jordan steps, respectively, have a simple form:

$$G^{(k)} := \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ c_{k,1} & c_{k,2} & \cdots & c_{k,k-1} & 1 \end{bmatrix}, \quad J^{(k)} := \begin{bmatrix} 1 & & & & d_{1,k} \\ & 1 & & & d_{2,k} \\ & & \ddots & & \vdots \\ & & & 1 & d_{k-1,k} \\ & & & & 1 \end{bmatrix}.$$

Therefore, the Gauss-Jordan elimination method can be easily implemented for row-finite matrices $\mathcal{A}$ with a finite number of rows.

Denote by $\mathcal{H}|_n$ the row-finite matrix in $\mathrm{RFM}_n(\mathbb{R})$ obtained from the first n rows of $\mathcal{H} \in \mathrm{RFM}_k(\mathbb{R})$. Similarly, denote by $Q|_n \in \mathbb{R}^{n\times k}$ the matrix obtained from the first n rows of $Q \in \mathbb{R}^{k\times k}$. Below, we have a stabilization result for the Gauss-Jordan Elimination method regarding row-finite matrices $\mathcal{A} \in \mathrm{RFM}(\mathbb{R})$ with countable number of rows.

Theorem 5.4.1 (Chain of row-finite matrices)**.** *Let $\mathcal{A} \in \mathrm{RFM}(\mathbb{R})$. Consider the sequence of matrices and row-finite matrices generated by the Gauss-Jordan algorithm. Given $n \geq 1$, there exists $\sigma_n \geq n$ such that*

$$\mathcal{H}^{(k)}\big|_n = \mathcal{H}^{(\sigma_n)}\big|_n, \qquad Q^{(k)}\big|_n = \left[Q^{(\sigma_n)}\big|_n \; 0\right],$$

for every $k \geq \sigma_n$, where $0 \in \mathbb{R}^{n\times(k-\sigma_n)}$ is a zero matrix. In other words, given $n \in \mathbb{N}$, the first n rows of $\mathcal{H}^{(k)}$ and $Q^{(k)}$ stabilizes after a finite number of iterations $\sigma_n \in \mathbb{N}$, where $\sigma_n \geq n$.

Proof. First, the length of any fixed row of $\mathcal{H}^{(k)}|_n$ is a non-increasing function of $k \geq n$. Indeed, the Gaussian step and the normalization step do not change the rows of $\mathcal{H}^{(k)}\Big|_n$ after the n-th iteration, and the Jordan step can only decrease the length of those rows. The permutation step only swaps a row by another one of shorter length. Hence, the length of any fixed row of $\mathcal{H}^{(k)}|_n$ is a non-increasing function of $k \geq n$.

This implies that the Gauss-Jordan elimination method does not swap the rows of $\mathcal{H}^{(k)}|_n$ after a finite number of iterations $\sigma_n \geq n$. In particular, the Jordan step at any iteration $k \geq \sigma_n$ does not change the rows of $\mathcal{H}^{(k)}|_n$ either, otherwise the length of h_k at the Jordan step would be strictly less than the length of some row of $\mathcal{H}^{(k)}|_n$ and the next permutation step would swap some row of $\mathcal{H}^{(k)}|_n$ by h_k, at an iteration $k \geq \sigma_n$.

For any iteration $k \geq \sigma_n$, the elementary matrix E_k becomes the block matrix

$$E_k = \begin{bmatrix} I_n & 0 \\ C_{k-n} & D_{k-n} \end{bmatrix}, \tag{5.11}$$

where $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix, $0 \in \mathbb{R}^{n \times (k-n)}$ is the zero matrix, $C_{k-n} \in \mathbb{R}^{(k-n)\times n}$, and $D_{k-n} \in \mathbb{R}^{(k-n)\times(k-n)}$. The recursive relation of $Q^{(k)}$ given by (5.10) and the block expression of E_k in Equation (5.11) imply that $Q^{(k)}|_n$ is equal to $\left[Q^{(\sigma_n)}\Big|_n \ 0\right]$, for any iteration $k \geq \sigma_n$. $\square$

An important consequence of Theorem 5.4.1 is the chain of row-finite matrices

$$\mathcal{H}^{(\sigma_1)}\Big|_1 \sqsubseteq \mathcal{H}^{(\sigma_2)}\Big|_2 \sqsubseteq \cdots \sqsubseteq \mathcal{H}^{(\sigma_n)}\Big|_n \sqsubseteq \cdots, \tag{5.12}$$

where the relation $\mathcal{H}^{(\sigma_i)}\Big|_i \sqsubseteq \mathcal{H}^{(\sigma_{i+1})}\Big|_{i+1}$ represents the equality $\mathcal{H}^{(\sigma_i)}\Big|_i = \mathcal{H}^{(\sigma_{i+1})}\Big|_i$. It follows from Theorem 5.4.1 that the row-finite matrix $\mathcal{H} \in \mathrm{RFM}(\mathbb{R})$ with *countable rows* and given by $\mathcal{H}|_n := \mathcal{H}^{(\sigma_n)}\Big|_n$, for all $n \in \mathbb{N}$, is well defined and in HNF. The matrix $Q^{(k)} \in \mathbb{R}^{k \times k}$ can be extended to a row-finite matrix with k rows, $\mathcal{Q}^{(k)} \in \mathrm{RFM}_k(\mathbb{R})$, by the natural immersion map. It also follows from Theorem 5.4.1 that the row-finite matrix $\mathcal{Q} \in \mathrm{RFM}(\mathbb{R})$ defined as $\mathcal{Q}|_n := \mathcal{Q}^{(\sigma_n)}\big|_n$ is well defined. The next result establishes the relation between $\mathcal{A}$, $\mathcal{H}$, and $\mathcal{Q}$.

Theorem 5.4.2. *The row-finite matrix $\mathcal{Q}$ is non-singular and $\mathcal{H} = \mathcal{Q}\mathcal{A}$.*

Proof. The proof of Theorem 5.4.2 is referred to [67]. □

5.5 An application to extreme flow characterization for infinite digraphs

In this section, we provide an alternative proof for the characterization theorem of extreme flows over infinite digraphs of finite node degrees. We revisit the result of [11] under the scope of row-finite linear systems and the characterization of Theorem 5.3.1 regarding basic feasible solutions.

Let $G = (\mathcal{N}, \mathcal{A})$ be a directed graph where the number of nodes $\mathcal{N}$ is countable and the number of arcs connected to each node is finite. The set of arcs $\mathcal{A}$ is also countable since it is a countable union of finite sets. Below, we recall some definitions for digraphs:

- We say that the digraph G is *weakly connected* if one replaces its arcs by undirected edges and the resulting graph is connected. See Figures 5.1a and 5.1b for an example of a weakly connected digraph.

- We say that C is a *weak cycle* of G if C is a cycle for the induced undirected graph. See Figure 5.1a for an example. We call a digraph G *acyclic* if it has no weak cycle.

- A *tree* T is a digraph that is weakly connected and has no weak cycle. See Figure 5.1b for an example of a tree.

- Let U be a subset of $\mathcal{N}$. We define the cut-set of *outgoing*, *incoming*, and *crossing* arcs as

$$\delta^+(U) = \{a \in \mathcal{A} \mid a = (i,j),\ i \in U,\ j \in \mathcal{N} \backslash U\}, \tag{5.13}$$

$$\delta^-(U) = \{a \in \mathcal{A} \mid a = (j,i),\ i \in U,\ j \in \mathcal{N} \backslash U\}, \tag{5.14}$$

$$\delta(U) = \delta^+(U) \cup \delta^-(U), \tag{5.15}$$

respectively. We abuse notation and denote by $\delta^+(i)$, $\delta^-(i)$, and $\delta(i)$ the cut-set of outgoing, incoming, and crossing arcs for the singleton $\{i\}$, respectively. We only consider digraphs G of finite degree, that is, the cardinality of $\delta(i)$ is finite, for every node $i \in \mathcal{N}$.

- We call the *network flow* set associated to the digraph G the convex set below:

$$P = \left\{ x \in \mathbb{R}^{\mathcal{A}} \;\middle|\; \begin{array}{cc} \sum_{a \in \delta^+(i)} x(a) - \sum_{\delta^-(i)} x(a) = d(i), & \forall i \in \mathcal{N}, \\ \underline{b}(a) \leq x(a) \leq \overline{b}(a), & \forall a \in \mathcal{A}. \end{array} \right\}, \tag{5.16}$$

  where $d : \mathcal{N} \to \mathbb{R}$ is the node demand function, and $\underline{b}, \overline{b} : \mathcal{A} \to \mathbb{R}$ are arc bound functions. The constraints that define P in (5.16) form a row-finite linear system.

- Given a flow $x \in P$, we call the *residual capacity* at an arc $a \in \mathcal{A}$ the quantity

$$r(a) = \min\{x(a) - \underline{b}(a),\ \overline{b}(a) - x(a)\}. \tag{5.17}$$

  It represents the maximum amount by which one can perturb the flow x at a while preserving the arc bounds $\underline{b}(a) \leq x(a) \leq \overline{b}(a)$. The function r in (5.17) is called the *residual function* induced by the network flow $x \in P$. A general residual function is any non-negative function with domain $\mathcal{A}$.

- We denote by $\mathcal{A}(x)$ the set of arcs $a \in \mathcal{A}$ with positive residual capacity, i.e., $r(a) > 0$. We call *residual graph* $G(x)$ the digraph induced by the arc subset $\mathcal{A}(x)$.

- We call the *max-residual flow* with source node $i \in \mathcal{N}$ the quantity

$$R(i,G) = \sup_{h,u} \quad h \tag{5.18a}$$

$$\text{s.t.} \quad \sum_{a \in \delta^+(k)} u(a) - \sum_{a \in \delta^-(k)} u(a) = h \cdot \epsilon_{ik}, \quad \forall k \in \mathcal{N}, \tag{5.18b}$$

$$-r(a) \leq u(a) \leq r(a), \qquad \forall a \in \mathcal{A}, \tag{5.18c}$$

$$u \in \mathbb{R}^{\mathcal{A}},\ h \in \mathbb{R}, \tag{5.18d}$$

  where ϵ_{ik} represents the Kronecker function, that is, $\epsilon_{ik} = 1$ if i equals k, and $\epsilon_{ik} = 0$ otherwise. Intuitively, $R(i,G)$ is the maximum amount by which one can perturb the node demand at i and expect the existence of a feasible flow.

- We call the *min-residual cut* with source node $i \in \mathcal{N}$ the quantity

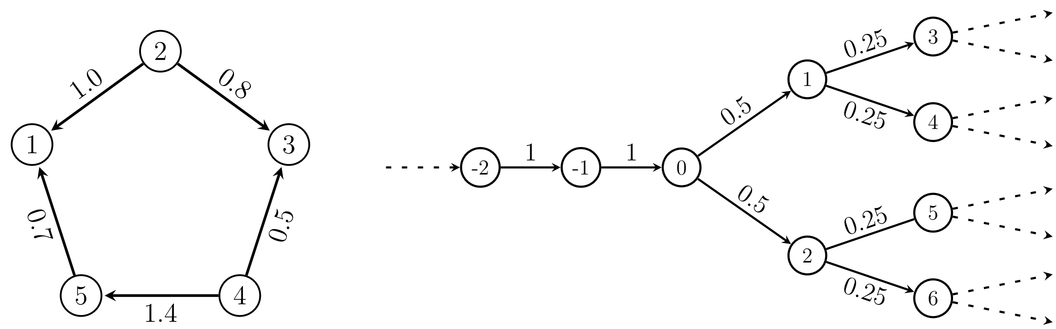
$$C(i,G) = \inf_{\substack{U \subseteq \mathcal{N}; \\ i \in U,\ |U| < \infty}} \sum_{a \in \delta(U)} r(a). \tag{5.19}$$

  It represents the residual bottleneck capacity at $i \in \mathcal{N}$ and can be interpreted similarly to the max-residual flow, see Lemma 5.5.2.

We now illustrate some important ideas regarding extreme flows over infinite digraphs of finite node degrees. Let G be an infinite digraph where the set of nodes is $\mathbb{Z}$ and it satisfies the finite degree property. Let the flow lower bound $\underline{b}(a)$ be 0 and the flow upper bound $\overline{b}(a)$ be 2, for every arc $a \in \mathcal{A}$. Let the demand $d(i)$ be 0, for every node $i \in \mathbb{Z}$.

We recall from basic network flow theory that if the residual graph $G(x)$ associated to a flow $x \in P$ has a weak cycle then x is not extreme. In Figure 5.1a, we illustrate a weak cycle in a residual graph $G(x)$, where the numbers in each arc represent the associated flow. When this happens, one can create two solutions by slightly increasing the total flow along the cycle in the clockwise and anti-clockwise directions. This crease two flows $x^1, x^2 \in P$ such that $(x^1 + x^2)/2 = x$, which implies that x is not an extreme flow.

However, a new phenomenon occurs for infinite digraphs: even if the residual graph $G(x)$ is acyclic, the flow x may not be extreme. Consider the residual graph in Figure 5.1b. One can create different solutions by increasing the total flow to the "left" or to the "right" of $G(x)$, then proving that x is a convex combination of two solutions. The extreme flow characterization theorem states the exact conditions for x to be an extreme point.



(a) Weak cycle in a residual graph $G(x)$.

(b) Tree in a residual graph $G(x)$.

Figure 5.1: Non-extreme flows in infinite digraphs with finite degree.

To understand the intuition of the second condition of the extreme point characterization, one needs an interpretation of the max-residual flow. Indeed, the max-residual flow $R(i, G)$ with source node $i \in \mathcal{N}$ is a less conventional type of max-flow problem since it *does not consider a sink node*. The quantity $R(i, G)$ represents the maximum additional flow one can inject or withdraw at $i \in \mathcal{N}$ and re-route it to or from "infinity". If G is a finite graph, then the quantity $R(i, G)$ is 0 since there is no sink node. One can also observe this property from the conservation equation (5.18b):

$$h = \sum_{k \in \mathcal{N}} \left(\sum_{a \in \delta^+(k)} u(a) - \sum_{\delta^-(k)} u(a) \right) = 0.$$

Besides no weak cycle in the residual graph $G(x)$, the other necessary condition is that there are no two arc-disjoint trees T, S in the residual graph $G(x)$ with positive max-residual flow $R(i, T), R(i, S) > 0$ at some common node $i \in \mathcal{N}_T \cap \mathcal{N}_S$. Those two conditions together are necessary and sufficient for a flow x to be an extreme point.

In order to prove this characterization, we need a lemma that guarantees the existence of a feasible solution to the row-finite linear system defined by the constraints of (5.18). In fact, such lemma is the the generalization of the max-flow min cut for a residual tree.

Indeed, let $T = (\mathcal{N}, \mathcal{A})$ be a tree with countable number of nodes and the finite degree property, and let $i \in \mathcal{N}$ be the root node of T. Let T_a be the subtree defined by the descendants of i in T after the arc a, and let $T_a \backslash \{i\}$ be the subtree T_a without the node i. Clearly, the tree T is the union of arc-disjoint subtrees T_a, for all $a \in \delta(i)$. The following lemma allows us to recursively create local solutions at arcs incident to each node $j \in \mathcal{N}$, and patch them together to form a feasible solution to the max-residual system (5.18).

Lemma 5.5.1 (Recursive local balance for residual escape tree)**.** *Let $T = (\mathcal{N}, \mathcal{A})$ be a tree with countable number of nodes and the finite degree property. Let r be a residual function at T and let $i \in \mathcal{N}$ be a node. If $|h| \leq C(i, T)$ then the linear system*

$$\sum_{a \in \delta^+(i)} u(a) - \sum_{a \in \delta^-(i)} u(a) = h, \tag{5.20a}$$

$$-r(a) \leq u(a) \leq r(a), \qquad \forall a \in \delta(i), \tag{5.20b}$$

has a solution $u \in \mathbb{R}^{\delta(i)}$ such that $|u(a)| \leq C(j_a, T_a \backslash \{i\})$, for every arc $a \in \delta(i)$, where j_a is the endpoint of the arc a different than i.

Proof. It follows from the arc-disjoint decomposition of T as union of T_a's and the min-residual cut definition (5.19) that

$$C(i, T) = \sum_{a \in \delta(i)} C(i, T_a). \tag{5.21}$$

Because the node i in T_a has only the arc a incident to it, we can represent the min-residual cut $C(i, T_a)$ as the minimum between the residual capacity $r(a)$ and the min-residual cut $C(j_a, T_a \backslash \{i\})$ for the tree $T_a \backslash \{i\}$ at source node j_a:

$$C(i, T_a) = \min\Big\{r(a),\ C(j_a, T_a\backslash\{i\})\Big\}. \tag{5.22}$$

We conclude the result by finding a solution $u \in \mathbb{R}^{\delta(i)}$ to the node balance equation (5.20a) such that $|u(a)| \le C(i, T_a)$, for every $a \in \delta(i)$. Indeed, if such solutions exists then it follows from (5.22) that u satisfies the arc bound (5.20b) and the desired condition $|u(a)| \le C(j_a, T_a\backslash\{i\})$, for every arc $a \in \delta(i)$.

Consider the continuous function $\Phi(u) := \sum_{a\in\delta^+(i)} u(a) - \sum_{a\in\delta^-(i)} u(a)$ and let K be the Cartesian product of intervals $\prod_{a\in\delta(a)} \Big[-C(i,T_a),\ C(i,T_a)\Big]$. Note that K is a connected and compact set, and the maximum of Φ over K has the form

$$\max_{u\in K} \Phi(u) = \sum_{a\in\delta(i)} C(i, T_a) = C(i, T).$$

We have a similar expression for the minimum of Φ over K, $\min_{u\in K} \Phi(u) = -C(i, T)$. This implies that the image of K by Φ is the interval $\Big[-C(i,T), C(i,T)\Big]$, which contains h, by hypothesis. Hence, there is a solution $u \in \mathbb{R}^{\delta(a)}$ to the node balance equation (5.20a) such that $|u(a)| \le C(i, T_a)$, for every $a \in \delta(i)$. □

Lemma 5.5.2 (Max-flow min-cut for residual escape tree)**.** *Let $T = (\mathcal{N}, \mathcal{A})$ be a tree with countable number of nodes and the finite degree property. Let r be a residual function at T and let $i \in \mathcal{N}$ be a node. Then, the max-residual system with source node $i \in \mathcal{N}$,*

$$\sum_{a\in\delta^+(k)} u(a) - \sum_{a\in\delta^-(k)} u(a) = h \cdot \epsilon_{ik}, \quad \forall k \in \mathcal{N}, \tag{5.23a}$$

$$-r(a) \le u(a) \le r(a), \qquad \forall a \in \mathcal{A}, \tag{5.23b}$$

has a feasible solution if, and only if, $|h| \le C(i, T)$. In particular, the max-residual flow $R(i, T)$ is equal to the min-residual cut $C(i, T)$, i.e., $R(i, T) = C(i, T)$.

Proof. Suppose the linear system (5.23) has a feasible solution $u \in \mathbb{R}^{|\mathcal{A}|}$. Given any subset $U \subseteq \mathcal{N}$ such that $|U| < \infty$ and $i \in U$, we have that

$$\begin{aligned} h &= \sum_{k \in U} \left(\sum_{a \in \delta^+(k)} u(a) - \sum_{a \in \delta^-(k)} u(a) \right) \\ &= \sum_{a \in \delta^+(U)} u(a) - \sum_{a \in \delta^-(U)} u(a), \end{aligned}$$

where the second equality comes from the fact that $u(a)$ cancels out in the expression inside the parenthesis if a is an arc that that has both endpoint in U. We then conclude that

$$\begin{aligned} |h| &\leq \sum_{a \in \delta^+(U)} |u(a)| + \sum_{a \in \delta^-(U)} |u(a)| \\ &\leq \sum_{a \in \delta^+(U)} r(a) + \sum_{a \in \delta^-(U)} r(a) = \sum_{a \in \delta(U)} r(a). \end{aligned}$$

Hence, the inequality $|h| \leq C(i, T)$ holds.

Conversely, suppose that $|h| \leq C(i, T)$. Starting from the root node i, it follows from Lemma 5.5.1 that we can inductively find local solutions $u \in \mathbb{R}^{\delta(j)}$ at arcs incident to a node $j \in \mathcal{N}$, use those solutions as input $h = u(a)$ for the linear system (5.20) at the adjacent nodes, and repeat the argument. This sequence of local solutions forms a global solution $u \in \mathbb{R}^{\mathcal{A}}$ to (5.23).

□

Now that we proved Lemma 5.5.2 about the max-flow min-cut for residual infinite tree, we prove the extreme flow characterization theorem.

Theorem 5.5.1 (Extreme flows characterization, [11])**.** *Let $G = (\mathcal{N}, \mathcal{A})$ be a digraph and let $x \in P$ be a network flow at G. Let r be the residual function induced by x. Then, x is an extreme point of P if, and only if, the residual graph $G(x)$:*

1. *has no weak cycle; and*

2. *has no arc-disjoint subtrees T and S with positive max-residual flow at a common node, that is, $R(i,T), R(i,S) > 0$, for some $i \in \mathcal{N}_T \cap \mathcal{N}_S$.*

Proof. Suppose that $x \in P$ is an extreme point. From Theorem 5.3.1, this is equivalent to say that the following row-finite system have only the zero vector as an AC-x solution:

$$\sum_{a\in\delta^+(k)} u(a) - \sum_{a\in\delta^-(k)} u(a) = 0, \qquad \forall k \in \mathcal{N}, \tag{5.24a}$$

$$u(a) = 0, \qquad \forall a \in \mathcal{A};\ r(a) = 0. \tag{5.24b}$$

Suppose the residual graph $G(x)$ has a cycle $L := i_1 i_2 \cdots i_n i_1$. We say that an arc a of L is oriented clockwise if $a = (i_t, i_{t+1})$ or $a = (i_n, i_1)$, and it is oriented anti-clockwise if $a = (i_{t+1}, i_t)$ or $a = (i_1, i_n)$, for some $t = 1, \ldots, n-1$. Let $u \in \mathbb{R}^{\mathcal{A}}$ be defined as

$$u(a) = \begin{cases} 1, & \text{if } a \text{ belongs to } L \text{ and it is oriented clockwise,} \\ -1, & \text{if } a \text{ belongs to } L \text{ and it is oriented anti-clockwise,} \\ 0, & \text{if } a \text{ does not belong to } L. \end{cases} \tag{5.25}$$

This is clearly a non-zero AC-x solution to (5.24). Hence, the condition 1 must hold.

Suppose the residual graph $G(x)$ has two arc-disjoint subtrees T and S with positive max-residual flow $R(i,T)$ and $R(i,S)$, for some common node $i \in \mathcal{N}_T \cap N_S$. Let h be the minimum value between $R(i,T)$ and $R(i,S)$. It follows from Lemma 5.5.2 that if we consider the input h and $-h$ for the linear system (5.23) regarding the trees T and S, respectively, then there exist non-zero solutions $u_T \in \mathbb{R}^{\mathcal{A}_T}$ and $u_S \in \mathbb{R}^{\mathcal{A}_S}$ associated to each linear system. Since T and S are arc-disjoint subtrees of $G(x)$, the following solution $u \in \mathbb{R}^{\mathcal{A}}$ is well-defined:

$$u(a) = \begin{cases} u_T(a), & \text{if } a \in \mathcal{A}_T, \\ u_S(a), & \text{if } a \in \mathcal{A}_S, \\ 0, & \text{if } a \in \mathcal{A}\backslash\big(\mathcal{A}_T \cup \mathcal{A}_T\big). \end{cases} \tag{5.26}$$

By construction, u is a non-zero solution to (5.24). Also, note that u is an AC-x vector:

$$\sup_{a\in\mathcal{A};\ r(a)>0} \frac{|u(a)|}{r(a)} = \sup_{a\in\mathcal{A}_T\cup\mathcal{A}_S} \frac{|u(a)|}{r(a)} \leq 1, \tag{5.27}$$

where the last inequality comes from the arc bounds (5.23b). Hence, condition 2 must hold. Conversely, suppose conditions 1 and 2 hold. Suppose that $u \in \mathcal{R}^{\mathcal{A}}$ is a non-zero AC-x solution to (5.24). Then, it follows from the node conversevation equation (5.24a) that there is a maximal subtree W of $G(x)$ such that

- $u(a)$ is non-zero for every arc $a \in \mathcal{A}_W$.
- W has a countable number of nodes and the degree of every node is at least 2.

Let $i \in \mathcal{N}_W$ be any node of W. We decompose W into two arc-disjoint subtrees T and S with a common node i. If we normalize u by $c := \sup_{a\in\mathcal{A}_W;r(a)>0} \frac{|u(a)|}{r(a)}$ then $u|_T$ and $u|_S$ are two feasible solutions to the max-residual flow problem (5.23). Hence, $R(i,T), R(i,S) > 0$, which is a contradiction. Hence, the only AC-x solution to (5.24) is the zero vector. Therefore, x is an extreme point. $\square$

CHAPTER 6
THESIS SUMMARY

6.1 The OPCF-EBF model

In Chapter 2, we proposed the Optimal Planning of Charging Facilities and Electric Bus Fleets (OPCF-EBF) for the optimal transition to entirely electric fleets regarding the practical needs of public transit agencies. As described in Section 2.4, our model considers a yearly planning horizon for charging infrastructure and fleet renewal investment with bus retirement targets, charging location, and budget constraints.

Section 2.4 also described a realistic operation model to assess the fleet operation and costs each year. We used the bus schedules informed through the General Transit Feed Specification (GTFS) file regarding each public transit system to extract information about the routes and associated bus demand for each hourly time interval. Our operation meets the bus demand using a mix of electric and conventional bus fleets along the transition.

The operation model also offered insights into our model's operation peaks and charging dynamics. It represents a stationary bus schedule for a regular weekday, e.g., Monday. We assumed a 24-hour operation is periodic and repeats throughout the year with the coupling constraints between the variables of the last and first 24 hours.

We proved in Section 2.5 that the computational complexity of our OPCF-EBF model is NP-Hard, and its proof is a polynomial-time reduction from the Uncapacitated Facility Location (UFL) problem. In practice, the OPCF-EBF proved to be a numerically challenging problem, as observed in Section 2.7. Even Gurobi could not find a solution with an average gap smaller than 52% within 4 hours of computation in a cluster with 86 cores. We proposed a scalable primal heuristic in Section 2.6 that accelerated the search for an excellent primal solution, outperforming Gurobi in most real cases.

We concluded Section 2.7 with insights about strategies and comments for electrifying the city of Atlanta and Boston. We also observed fleet-sizing and operation patterns in the analysis of 11 other US cities and 2 non-US cities.

6.2 Analysis of a single route fleet sizing model

In Chapter 3, we investigated whether the scheduling of the bus charging, operation, and fleet sizing without charging location could be another cause for numerical issues. We analyzed in Section 3.3 the simplest fleet sizing and operation model, which assumes only one route, unlimited charging capacity, and depot BEBs. In terms of operation, the depot BEBs could only charge when depleted. Once fully recharged, they had to resume operation immediately. We proved that the fleet sizing model is polynomially solvable, but the proof of such a fact is nontrivial. Indeed, we reformulated in Section 3.3 our problem as a two-stage model, in which the first stage contains only the fleet-sizing variables and constraints. In contrast, the second stage is the operation, given the fleet size. We proved in Section 3.3.1 that the second stage problem is integral, i.e., the linear relaxation is the convex hull of the feasible integral set, despite the second stage problem's constraint matrix not being totally unimodular.

In Section 3.3.2, we framed our two-stage integer program as a Separable Convex Integral Program (SCIP). The novelty of our polynomial-time reduction lies in using a proximity theorem [5] for SCIPs to limit the search for an optimal integral solution. This analysis only works for instances where we do not allow either idle buses or early bus charging. For more flexible operations, the computational complexity remains open. Then, in Section 3.4, we generalized our model to a class of two-stage Separable Integral Programs. In Section 3.4.1, we introduced the Dyadic Contiguous Row (DCR) matrix that generalized the notion of a row circular matrix [6] which contains our second-stage constraint matrix as a particular case. Our polynomial-time algorithm is based on the proximity theorem for SCIPs and can also solve this new class of mixed-integer programs.

6.3 Some properties of stationary infinite-dimensional linear programs

In Chapter 4, motivated by the stationary operation of our bus schedules, we investigated the meaning of a stationary linear program more deeply. More precisely, we departed in Section 4.4 from a fixed point value function perspective and introduced the elements of a stationary infinite-dimensional linear program. The first challenge was to guarantee convergence of the discounted series that naturally arises in the objective function. One could take several approaches to make the objective well-defined, such as taking the series's liminf [7] or assuming a uniform bound for the decision variables [8, 9]. Indeed, we chose a balance between those two. We introduced the ℓ_∞ set constraint in defining a stationary infinite-dimensional linear program. Our approach preserves the objective's linearity property and is less restrictive than the uniform bounds on the variable space.

Following this analogy and still in Section 4.4, we introduced a dual stationary infinite-dimensional linear program. We applied the same duality rules as in a finite-dimensional linear program and added the ℓ_1 set constraint. Weak Duality follows from simple algebraic manipulations and Fubini's theorem for absolutely convergent series.

In Section 4.5.1, we provided a toy problem inspired by a hydro-thermal power generation planning problem that supports our primal-dual setting and satisfies Strong Duality. We observed in Section 4.5.2 through a counter-example that by dropping the ℓ_∞ and ℓ_1 set constraints Weak Duality may fail. However, once we enforced those set constraints, the same example satisfies Strong Duality. Those pieces of evidence support the claim that Strong Duality may hold for a large class of stationary infinite-dimensional linear programs.

Lastly, in Section 4.5.3, we connected our Hydro-Thermal power planning model with an infinite-horizon Lot-Sizing program from [56] and observed some conjectures about possible generalizations to periodic problems instead of stationary infinite-dimensional linear programs.

6.4 An extension of basic feasible solutions to infinite-dimensional programs

Since we obtained explicit primal and dual optimal solutions for the stationary Hydro-Thermal planning problem, the natural follow-up question we investigated in Chapter 5 was whether or not those solutions are extreme points. This motivated our algebraic characterization of extreme points in Section 5.3 as a more direct criterion to check if a given solution is extreme or not. We introduced the notion of an asymptotically compatible (AC) vector, which connects with the idea of a feasible direction. Indeed, we proved that a direction is AC with a feasible solution if, and only if, the direction and the opposite sign direction are both feasible at the same point. The asymptotically compatible concept was central to extend the notion of a basic feasible solution to any convex set defined by arbitrary linear constraints.

Using our extreme point characterization, we proved in Section 5.3.1 that the primal and dual optimal solutions to the hydro-thermal planning problems are extreme points. We then described in Section 5.4 a general method, called the Gauss-Jordan method, to find all the solutions for binding linear equality systems called the row-finite linear systems. Finally, we illustrated this technique in an extreme point example.

As a more general application, we provided in Section 5.5 an alternative proof of the extreme point characterization for network flows on infinite graphs of finite degrees [11]. We proved that a flow is extreme if, and only if, the residual graph has no weak cycle and there are not two arc-disjoint trees with positive max-residual capacity at a common node. Intuitively, we cannot reroute flows from "infinity" from one tree into another. Our extension of a basic feasible solution is a simplifying tool for the original extreme flow characterization of [11].

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